

# Nash-type equilibria on Riemannian manifolds: a variational approach

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## Abstract

Motivated by Nash equilibrium problems on 'curved' strategy sets, the concept of Nash-Stampacchia equilibrium points is introduced via variational inequalities on Riemannian manifolds. Characterizations, existence, and stability of Nash-Stampacchia equilibria are studied when the strategy sets are compact/noncompact geodesic convex subsets of Hadamard manifolds, exploiting two well-known geometrical features of these spaces both involving the metric projection map. These properties actually characterize the non-positivity of the sectional curvature of complete and simply connected Riemannian spaces, delimiting the Hadamard manifolds as the optimal geometrical framework of Nash-Stampacchia equilibrium problems. Our analytical approach exploits various elements from set-valued and variational analysis, dynamical systems, and non-smooth calculus on Riemannian manifolds. Examples are presented on the Poincaré upper-plane model and on the open convex cone of symmetric positive definite matrices endowed with the trace-type Killing form.

## Résumé

Motivé par des problèmes d'équilibres de Nash sur des ensembles "courbés" de stratégies, la notion d'équilibre de Nash-Stampacchia peut être introduite par des inégalités variationnelles sur des variétés Riemanniennes. On étudie la caractérisation, l'existence et la stabilité d'équilibres de Nash-Stampacchia quand les ensembles de stratégies sont des sous-ensembles géodésiquement convexes, compacts ou non-compacts, de variétés d'Hadamard, en exploitant deux propriétés géométriques bien connues de ces espaces, basées sur la projection métrique. En fait ces propriétés caractérisent la non-positivité de la courbure sectionnelle des espaces de Riemann complets et simplement connexes, en identifiant les variétés d'Hadamard comme la structure géométrique optimale où poser les problèmes d'équilibre de Nash-Stampacchia. Notre approche analytique utilise plusieurs éléments d'analyse variationnelle et multi-valuée, des systèmes dynamiques et le calcul non-lisse sur les variétés Riemanniennes. Des exemples sont présentés dans le cadre du demi-plan de Poincaré et dans le cône ouvert convexe des matrices définies positives muni d'une forme Killing de type trace.

*Key words:* Nash-Stampacchia equilibrium point, Riemannian manifold, metric projection, non-smooth analysis, non-positive curvature.

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## 1 Introduction

After the seminal paper of Nash [30] there has been considerable interest in the theory of Nash equilibria due to its applicability in various real-life phenomena (game theory, price theory, networks, etc). Appreciating Nash's contributions, R. B. Myerson states that "Nash's theory of noncooperative games should now be recognized as one of the outstanding intellectual advances of the twentieth century", see also [29]. The Nash equilibrium problem involves  $n$  players such that each player know the equilibrium strategies of the partners, but moving away from his/her own strategy alone a player has nothing to gain. Formally, if the sets  $K_i$  denote the strategies of the players and  $f_i : K_1 \times \dots \times K_n \rightarrow \mathbf{R}$  are their loss-functions,  $i \in \{1, \dots, n\}$ , the objective is to find an  $n$ -tuple  $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{K} = K_1 \times \dots \times K_n$  such that

$$f_i(\mathbf{p}) = f_i(p_i, \mathbf{p}_{-i}) \leq f_i(q_i, \mathbf{p}_{-i}) \text{ for every } q_i \in K_i \text{ and } i \in \{1, \dots, n\},$$

where  $(q_i, \mathbf{p}_{-i}) = (p_1, \dots, p_{i-1}, q_i, p_{i+1}, \dots, p_n) \in \mathbf{K}$ . Such point  $\mathbf{p}$  is called a *Nash equilibrium point* for  $(\mathbf{f}, \mathbf{K}) = (f_1, \dots, f_n; K_1, \dots, K_n)$ , the set of these points being denoted by  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ .

While most of the known developments in the Nash equilibrium theory deeply exploit the usual convexity of the sets  $K_i$  together with the vector space structure of their ambient spaces  $M_i$  (i.e.,  $K_i \subset M_i$ ), it is nevertheless true that these results are in large part *geometrical* in nature. The main purpose of this paper is to enhance those geometrical

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and analytical structures which serve as a basis of a systematic study of location of Nash-type equilibria in a general setting as presently possible. In the light of these facts our contribution to the Nash equilibrium theory should be considered intrinsical and analytical rather than game-theoretical. However, it seems that some ideas of the present paper can be efficiently applied to evolutionary game dynamics on curved spaces, see Bomze [5], and Hofbauer and Sigmund [17].

Before to start describing our results, we point out an important (but neglected) topological achievement of Ekeland [15] concerning the *existence* of Nash-type equilibria for two-person games on compact manifolds based on transversality and fixed point arguments. Without the sake of completeness, Ekeland's result says that if  $f_1, f_2 : M_1 \times M_2 \rightarrow \mathbf{R}$  are continuous functions having also some differentiability properties where  $M_1$  and  $M_2$  are compact manifolds whose Euler-Poincaré characteristics are non-zero (orientable case) or odd (non-orientable case), then there exists at least a Nash-type equilibria for  $(f_1, f_2; M_1, M_2)$  formulated via first order conditions involving the terms  $\frac{\partial f_i}{\partial x_i}$ ,  $i = 1, 2$ .

In the present paper we assume *a priori* that the strategy sets  $K_i$  are *geodesic convex* subsets of certain finite-dimensional Riemannian manifolds  $(M_i, g_i)$ , i.e., for any two points of  $K_i$  there exists a unique geodesic in  $(M_i, g_i)$  connecting them which belongs entirely to  $K_i$ . This approach can be widely applied when the strategy sets are 'curved'. Note that the choice of such Riemannian structures does not influence the Nash equilibrium points for  $(\mathbf{f}, \mathbf{K})$ . As far as we know, the first step into this direction was made recently in [20] via a McClendon-type minimax inequality for acyclic ANRs, guaranteeing the existence of at least one Nash equilibrium point for  $(\mathbf{f}, \mathbf{K})$  whenever  $K_i \subset M_i$  are compact and geodesic convex sets of certain finite-dimensional Riemannian manifolds  $(M_i, g_i)$  while the functions  $f_i$  have certain regularity properties,  $i \in \{1, \dots, n\}$ . By using Clarke-calculus on manifolds, in [20] we introduced and studied for a wide class of *non-smooth* functions the set of *Nash-Clarke equilibrium points* for  $(\mathbf{f}, \mathbf{K})$ , denoted in the sequel as  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ ; see Section 3. Note that  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$  is larger than  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ ; thus, a promising way to localize the elements of  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$  is to determine the set  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$  and to choose among these points the appropriate ones. In spite of the naturalness of this approach, we already pointed out its limited applicability due to the involved definition of  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ , conjecturing a more appropriate concept in order to locate the elements of  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ .

Motivated by the latter problem, we observe that the Fréchet and limiting subdifferential calculus of lower semicontinuous functions on Riemannian manifolds developed by Azagra, Ferrera and López-Mesas [1], and Ledyaeu and Zhu [23] provides a satisfactory approach. The idea is to consider the following *system of variational inequalities*: find  $\mathbf{p} \in \mathbf{K}$  and  $\xi_C^i \in \partial_C^i f_i(\mathbf{p})$  such that

$$\langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0 \text{ for all } q_i \in K_i, \ i \in \{1, \dots, n\},$$

where  $\partial_C^i f_i(\mathbf{p})$  denotes the Clarke subdifferential of the locally Lipschitz function  $f_i(\cdot, \mathbf{p}_{-i})$  at the point  $p_i \in K_i$ ; for details, see Section 3. The solutions of this system form the set

of *Nash-Stampacchia*<sup>3</sup> equilibrium points for  $(\mathbf{f}, \mathbf{K})$ , denoted by  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ , which is the main concept of the present paper.

Our first result shows that

$$\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$$

for the same class of non-smooth functions  $\mathbf{f} = (f_1, \dots, f_n)$  as in [20] (see Theorem 3.1 (i)). Although  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$  and  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$  coincide, the set of Nash-Stampacchia equilibrium points is more flexible and applicable. Indeed, via this new notion we can handle both compact and non-compact strategy sets, and we identify the geometric framework where this argument works, i.e., the class of Hadamard manifolds. These kinds of results could not have been achieved via the notion of Nash-Clarke equilibrium points as we will describe later.

To establish the above inclusions we introduce a notion of subdifferential for non-smooth functions by means of the cut locus which could be of some interest in its own right as well. Then, explicit characterizations of the Fréchet and limiting normal cones of geodesic convex sets in arbitrarily Riemannian manifolds are given by exploiting some fundamental results from [1] and [23]. If  $\mathbf{f} = (f_1, \dots, f_n)$  verifies a suitable 'diagonal' convexity assumption then we have equalities in the above relation (see Theorem 3.1 (ii)).

Having these inclusions in mind, the main purpose of the present paper is to establish existence, location and stability of Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  in different settings. While a Nash equilibrium point is obtained precisely as the fixed point of a suitable function (see for instance Nash's original proof via Kakutani fixed-point theorem), Nash-Stampacchia equilibrium points are expected to be characterized in a similar way as fixed points of a special map defined on the product Riemannian manifold  $\mathbf{M} = M_1 \times \dots \times M_n$  endowed with its natural Riemannian metric  $\mathbf{g}$  inherited from the metrics  $g_i$ ,  $i \in \{1, \dots, n\}$ . In order to achieve this aim, certain curvature and topological restrictions are needed on the manifolds  $(M_i, g_i)$ . By assuming that the ambient Riemannian manifolds  $(M_i, g_i)$  for the geodesic convex strategy sets  $K_i$  are *Hadamard manifolds*, our key observation (see Theorem 4.1) is that  $\mathbf{p} \in \mathbf{K}$  is a Nash-Stampacchia equilibrium point for  $(\mathbf{f}, \mathbf{K})$  if and only if  $\mathbf{p}$  is a fixed point of the set-valued map  $A_\alpha^\mathbf{f} : \mathbf{K} \rightarrow 2^\mathbf{K}$  defined by

$$A_\alpha^\mathbf{f}(\mathbf{p}) = P_\mathbf{K}(\exp_\mathbf{p}(-\alpha \partial_C^\Delta \mathbf{f}(\mathbf{p}))).$$

Here,  $P_\mathbf{K}$  is the metric projection operator associated to the geodesic convex set  $\mathbf{K} \subset \mathbf{M}$ ,  $\alpha > 0$  is a fixed number, and  $\partial_C^\Delta \mathbf{f}(\mathbf{p})$  denotes the diagonal Clarke subdifferential at point  $\mathbf{p}$  of  $\mathbf{f} = (f_1, \dots, f_n)$ ; see Section 3.

Within this geometrical framework, two cases are discussed. On the one hand, when  $\mathbf{K} \subset \mathbf{M}$  is *compact*, one can prove via the Begle's fixed point theorem for set-valued maps the existence of at least one Nash-Stampacchia equilibrium point for  $(\mathbf{f}, \mathbf{K})$  (see Theorem

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<sup>3</sup> Terminology introduced in honor of G. Stampacchia for his deep contributions to the theory of variational inequalities.

4.2). On the other hand, when  $\mathbf{K} \subset \mathbf{M}$  is *not* necessarily *compact*, we provide two types of results. First, based on a suitable coercivity assumption on  $\partial_C^\Delta \mathbf{f}$ , combined with the result for compact sets, we are able to guarantee the existence of at least one Nash-Stampacchia equilibrium point for  $(\mathbf{f}, \mathbf{K})$  (see Theorem 4.3). Second, by requiring more regularity on  $\mathbf{f}$  in order to avoid technicalities, we consider two dynamical systems; a discrete one

$$(DDS)_\alpha \quad \mathbf{p}_{k+1} = A_\alpha^\mathbf{f}(P_\mathbf{K}(\mathbf{p}_k)), \quad \mathbf{p}_0 \in \mathbf{M};$$

and a continuous one

$$(CDS)_\alpha \quad \begin{cases} \dot{\eta}(t) = \exp_{\eta(t)}^{-1}(A_\alpha^\mathbf{f}(P_\mathbf{K}(\eta(t)))) \\ \eta(0) = \mathbf{p}_0 \in \mathbf{M}. \end{cases}$$

By assuming a Lipschitz-type condition on  $\partial_C^\Delta \mathbf{f}$ , one can prove that the set of Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  is a *singleton* and the orbits of both dynamical systems exponentially converge to this unique point (see Theorem 4.4). Here, we exploit some arguments from the theory of differential equations on manifolds as well as careful comparison results of Rauch-type. It is clear by construction that the orbit of  $(DDS)_\alpha$  is viable relative to the set  $\mathbf{K}$ , i.e.,  $\mathbf{p}_k \in \mathbf{K}$  for every  $k \geq 1$ . By using a recent result of Ledyaev and Zhu [23], one can also prove an invariance property of the set  $\mathbf{K}$  with respect to the orbit of  $(CDS)_\alpha$ . Note that the aforementioned results concerning the 'projected' dynamical system  $(CDS)_\alpha$  are new even in the Euclidean setting; see Cavazzuti, Pappalardo and Passacantando [8], Xia [35], and Xia and Wang [36].

Since the manifolds  $(M_i, g_i)$  are assumed to be of Hadamard type (see Theorems 4.1-4.4), so is the product manifold  $(\mathbf{M}, \mathbf{g})$ . Our analytical arguments deeply exploit two geometrical features of the product *Hadamard manifold*  $(\mathbf{M}, \mathbf{g})$  concerning the metric projection operator for closed, geodesic convex sets:

- (A) *Validity of the obtuse-angle property*, see Proposition 2.1 (i). This fact is exploited in the characterization of Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  via the fixed points of the map  $A_\alpha^\mathbf{f}$ , see Theorem 4.1.
- (B) *Non-expansiveness of the projection operator*, see Proposition 2.1 (ii). This property is applied several times in the proof of Theorems 4.2 and 4.4.

It is natural to ask to what extent the Riemannian structures of  $(M_i, g_i)$  are determined when the properties (A) and (B) simultaneously hold on the product manifold  $(\mathbf{M}, \mathbf{g})$ . A constructive proof combined with the formula of sectional curvature via the Levi-Civita parallelogramoid and a result of Chen [10] shows that if  $(M_i, g_i)$  are complete, simply connected Riemannian manifolds then (A) and (B) are simultaneously verified on  $(\mathbf{M}, \mathbf{g})$  if and only if  $(M_i, g_i)$  are Hadamard manifolds (see Theorem 5.1). Consequently, we may assert that Hadamard manifolds are the optimal geometrical framework to elaborate a fruitful theory of Nash-Stampacchia equilibrium problems on Riemannian manifolds. Furthermore, we notice that properties (A) and (B) are also the milestones of the theory of monotone vector fields, proximal point algorithms and variational inequalities developed on Hadamard manifolds by Li, López and Martín-Márquez [24], [25], and Németh [31].

As a byproduct of Theorem 5.1 we state that Hadamard manifolds are the appropriate geometrical frameworks among Riemannian manifolds for the aforementioned theories.

The paper is divided as follows. In §2 we recall/prove those notions and results which will be used throughout the paper: basic elements from Riemannian geometry, the parallelogramoid of Levi-Civita; properties of the metric projection; non-smooth calculus, dynamical systems and viability results on Riemannian manifolds. In §3 we compare the three Nash-type equilibria; simultaneously, we also recall some results from [20]. In §4, we prove the main results of this paper both for compact and non-compact strategy sets which are 'embedded' into certain Hadamard manifolds. In §5 we characterize the geometric properties (A) and (B) on  $(\mathbf{M}, \mathbf{g})$  by the Hadamard structures of the complete and simply connected Riemannian manifolds  $(M_i, g_i)$ ,  $i \in \{1, \dots, n\}$ . Finally, in §6 we present some relevant examples on the Poincaré upper-plane model as well as on the Hadamard manifold of symmetric positive definite matrices endowed with the Killing form of trace-type. Our examples are motivated by some applications from Bento, Ferreira and Oliveira [3], [4], Colao, López, Marino and Martín-Márquez [13], and Li and Yao [26].

## 2 Preliminaries: metric projections, non-smooth calculus and dynamical systems on Riemannian manifolds

**2.1. Elements from Riemannian geometry.** We first recall those elements from Riemannian geometry which will be used throughout the paper. We mainly follow Cartan [7] and do Carmo [14].

In this subsection,  $(M, g)$  is a connected  $m$ -dimensional Riemannian manifold,  $m \geq 2$ . Let  $TM = \cup_{p \in M} (p, T_p M)$  and  $T^*M = \cup_{p \in M} (p, T_p^* M)$  be the tangent and cotangent bundles to  $M$ . For every  $p \in M$ , the Riemannian metric induces a natural Riesz-type isomorphism between the tangent space  $T_p M$  and its dual  $T_p^* M$ ; in particular, if  $\xi \in T_p^* M$  then there exists a unique  $W_\xi \in T_p M$  such that

$$\langle \xi, V \rangle_{g,p} = g_p(W_\xi, V) \text{ for all } V \in T_p M. \quad (1)$$

Instead of  $g_p(W_\xi, V)$  and  $\langle \xi, V \rangle_{g,p}$  we shall write simply  $g(W_\xi, V)$  and  $\langle \xi, V \rangle_g$  when no confusion arises. Due to (1), the elements  $\xi$  and  $W_\xi$  are identified. With the above notations, the norms on  $T_p M$  and  $T_p^* M$  are defined by

$$\|\xi\|_g = \|W_\xi\|_g = \sqrt{g(W_\xi, W_\xi)}.$$

The generalized Cauchy-Schwartz inequality is also valid, i.e., for every  $V \in T_p M$  and  $\xi \in T_p^* M$ ,

$$|\langle \xi, V \rangle_g| \leq \|\xi\|_g \|V\|_g. \quad (2)$$

Let  $\xi_k \in T_{p_k}^*M$ ,  $k \in \mathbf{N}$ , and  $\xi \in T_p^*M$ . The sequence  $\{\xi_k\}$  converges to  $\xi$ , denoted by  $\lim_k \xi_k = \xi$ , when  $p_k \rightarrow p$  and  $\langle \xi_k, W(p_k) \rangle_g \rightarrow \langle \xi, W(p) \rangle_g$  as  $k \rightarrow \infty$ , for every  $C^\infty$  vector field  $W$  on  $M$ .

Let  $h : M \rightarrow \mathbf{R}$  be a  $C^1$  functional at  $p \in M$ ; the differential of  $h$  at  $p$ , denoted by  $dh(p)$ , belongs to  $T_p^*M$  and is defined by

$$\langle dh(p), V \rangle_g = g(\text{grad}h(p), V) \text{ for all } V \in T_pM.$$

If  $(x^1, \dots, x^m)$  is the local coordinate system on a coordinate neighborhood  $(U_p, \psi)$  of  $p \in M$ , and the local components of  $dh$  are denoted  $h_i = \frac{\partial h}{\partial x_i}$ , then the local components of  $\text{grad}h$  are  $h^i = g^{ij}h_j$ . Here,  $g^{ij}$  are the local components of  $g^{-1}$ .

Let  $\gamma : [0, r] \rightarrow M$  be a  $C^1$  path,  $r > 0$ . The length of  $\gamma$  is defined by

$$L_g(\gamma) = \int_0^r \|\dot{\gamma}(t)\|_g dt.$$

For any two points  $p, q \in M$ , let

$$d_g(p, q) = \inf\{L_g(\gamma) : \gamma \text{ is a } C^1 \text{ path joining } p \text{ and } q \text{ in } M\}.$$

The function  $d_g : M \times M \rightarrow \mathbf{R}$  is a metric which generates the same topology on  $M$  as the underlying manifold topology. For every  $p \in M$  and  $r > 0$ , we define the open ball of center  $p \in M$  and radius  $r > 0$  by

$$B_g(p, r) = \{q \in M : d_g(p, q) < r\}.$$

Let us denote by  $\nabla$  the unique natural covariant derivative on  $(M, g)$ , also called the Levi-Civita connection. A vector field  $W$  along a  $C^1$  path  $\gamma$  is called parallel when  $\nabla_{\dot{\gamma}}W = 0$ . A  $C^\infty$  parameterized path  $\gamma$  is a geodesic in  $(M, g)$  if its tangent  $\dot{\gamma}$  is parallel along itself, i.e.,  $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ . The geodesic segment  $\gamma : [a, b] \rightarrow M$  is called minimizing if  $L_g(\gamma) = d_g(\gamma(a), \gamma(b))$ .

Standard ODE theory implies that for every  $V \in T_pM$ ,  $p \in M$ , there exists an open interval  $I_V \ni 0$  and a unique geodesic  $\gamma_V : I_V \rightarrow M$  with  $\gamma_V(0) = p$  and  $\dot{\gamma}_V(0) = V$ . Due to the 'homogeneity' property of the geodesics (see [14, p. 64]), we may define the exponential map  $\exp_p : T_pM \rightarrow M$  as  $\exp_p(V) = \gamma_V(1)$ . Moreover,

$$d\exp_p(0) = \text{id}_{T_pM}. \tag{3}$$

Note that there exists an open (starlike) neighborhood  $\mathcal{U}$  of the zero vectors in  $TM$  and an open neighborhood  $\mathcal{V}$  of the diagonal  $M \times M$  such that the exponential map  $V \mapsto \exp_{\pi(V)}(V)$  is smooth and the map  $\pi \times \exp : \mathcal{U} \rightarrow \mathcal{V}$  is a diffeomorphism, where  $\pi$  is the canonical projection of  $TM$  onto  $M$ . Moreover, for every  $p \in M$  there exists

a number  $r_p > 0$  and a neighborhood  $\tilde{U}_p$  such that for every  $q \in \tilde{U}_p$ , the map  $\exp_q$  is a  $C^\infty$  diffeomorphism on  $B(0, r_p) \subset T_q M$  and  $\tilde{U}_p \subset \exp_q(B(0, r_p))$ ; the set  $\tilde{U}_p$  is called a *totally normal neighborhood* of  $p \in M$ . In particular, it follows that every two points  $q_1, q_2 \in \tilde{U}_p$  can be joined by a minimizing geodesic of length less than  $r_p$ . Moreover, for every  $q_1, q_2 \in \tilde{U}_p$  we have

$$\|\exp_{q_1}^{-1}(q_2)\|_g = d_g(q_1, q_2). \quad (4)$$

The *tangent cut locus* of  $p \in M$  in  $T_p M$  is the set of all vectors  $v \in T_p M$  such that  $\gamma(t) = \exp_p(tv)$  is a minimizing geodesic for  $t \in [0, 1]$  but fails to be minimizing for  $t \in [0, 1 + \varepsilon)$  for each  $\varepsilon > 0$ . The *cut locus* of  $p \in M$ , denoted by  $C_p$ , is the image of the tangent cut locus of  $p$  via  $\exp_p$ . Note that any totally normal neighborhood of  $p \in M$  is contained into  $M \setminus C_p$ .

We conclude this subsection by recalling a less used form of the *sectional curvature* by the so-called Levi-Civita parallelogramoid. Let  $p \in M$  and  $V_0, W_0 \in T_p M$  two vectors with  $g(V_0, W_0) = 0$ . Let  $\sigma : [-\delta, 2\delta] \rightarrow M$  be the geodesic segment  $\sigma(t) = \exp_p(tV_0)$  and  $W$  be the unique parallel vector field along  $\sigma$  with the initial data  $W(0) = W_0$ , the number  $\delta > 0$  being small enough. For any  $t \in [0, \delta]$ , let  $\gamma_t : [0, \delta] \rightarrow M$  be the geodesic  $\gamma_t(u) = \exp_{\sigma(t)}(uW(t))$ . The sectional curvature of the subspace  $S = \text{span}\{W_0, V_0\} \subset T_p M$  at the point  $p \in M$  is given by

$$K_p(S) = \lim_{u, t \rightarrow 0} \frac{d_g^2(p, \sigma(t)) - d_g^2(\gamma_0(u), \gamma_t(u))}{d_g(p, \gamma_0(u)) \cdot d_g(p, \sigma(t))},$$

see Cartan [7, p. 244-245]. The infinitesimal geometrical object determined by the four points  $p, \sigma(t), \gamma_0(u), \gamma_t(u)$  (with  $t, u$  small enough) is called the parallelogramoid of Levi-Civita.

**2.2. Metric projections.** Let  $(M, g)$  be an  $m$ -dimensional Riemannian manifold ( $m \geq 2$ ),  $K \subset M$  be a non-empty set. Let

$$P_K(q) = \{p \in K : d_g(q, p) = \inf_{z \in K} d_g(q, z)\}$$

be the set of *metric projections* of the point  $q \in M$  to the set  $K$ . Due to the theorem of Hopf-Rinow, if  $(M, g)$  is complete, then any closed set  $K \subset M$  is *proximal*, i.e.,  $P_K(q) \neq \emptyset$  for all  $q \in M$ . In general,  $P_K$  is a set-valued map. When  $P_K(q)$  is a singleton for every  $q \in M$ , we say that  $K$  is a *Chebyshev set*. The map  $P_K$  is *non-expansive* if

$$d_g(p_1, p_2) \leq d_g(q_1, q_2) \quad \text{for all } q_1, q_2 \in M \text{ and } p_1 \in P_K(q_1), p_2 \in P_K(q_2).$$

In particular,  $K$  is a Chebyshev set whenever the map  $P_K$  is non-expansive.

The set  $K \subset M$  is *geodesic convex* if every two points  $q_1, q_2 \in K$  can be joined by a unique minimizing geodesic whose image belongs to  $K$ . Note that (4) is also valid for



every  $q_1, q_2 \in K$  in a geodesic convex set  $K$  since  $\exp_{q_i}^{-1}$  is well-defined on  $K$ ,  $i \in \{1, 2\}$ . The function  $f : K \rightarrow \mathbf{R}$  is *convex*, if  $f \circ \gamma : [0, 1] \rightarrow \mathbf{R}$  is convex in the usual sense for every geodesic  $\gamma : [0, 1] \rightarrow K$  provided that  $K \subset M$  is a geodesic convex set.

A non-empty closed set  $K \subset M$  verifies the *obtuse-angle property* if for fixed  $q \in M$  and  $p \in K$  the following two statements are equivalent:

- (OA<sub>1</sub>)  $p \in P_K(q)$ ;
- (OA<sub>2</sub>) If  $\gamma : [0, 1] \rightarrow M$  is the unique minimal geodesic from  $\gamma(0) = p \in K$  to  $\gamma(1) = q$ , then for every geodesic  $\sigma : [0, \delta] \rightarrow K$  ( $\delta \geq 0$ ) emanating from the point  $p$ , we have  $g(\dot{\gamma}(0), \dot{\sigma}(0)) \leq 0$ .

**Remark 2.1** (a) In the Euclidean case  $(\mathbf{R}^m, \langle \cdot, \cdot \rangle_{\mathbf{R}^m})$ , (here,  $\langle \cdot, \cdot \rangle_{\mathbf{R}^m}$  is the standard inner product in  $\mathbf{R}^m$ ), every non-empty closed convex set  $K \subset \mathbf{R}^m$  verifies the obtuse-angle property, see Moskovitz-Dines [28], which reduces to the well-known geometric form:

$$p \in P_K(q) \Leftrightarrow \langle q - p, z - p \rangle_{\mathbf{R}^m} \leq 0 \text{ for all } z \in K.$$

(b) The first variational formula of Riemannian geometry shows that (OA<sub>1</sub>) implies (OA<sub>2</sub>) for every closed set  $K \subset M$  in a complete Riemannian manifold  $(M, g)$ . However, the converse does not hold in general; for a detailed discussion, see Kristály, Rădulescu and Varga [21].

A Riemannian manifold  $(M, g)$  is a *Hadamard manifold* if it is complete, simply connected and its sectional curvature is non-positive. It is well-known that on a Hadamard manifold  $(M, g)$  every geodesic convex set is a Chebyshev set, see Jost [18]. Moreover, we have

**Proposition 2.1** *Let  $(M, g)$  be a finite-dimensional Hadamard manifold,  $K \subset M$  be a closed set. The following statements hold true:*

- (i) (Walter [34]) *If  $K \subset M$  is geodesic convex, it verifies the obtuse-angle property;*
- (ii) (Grognet [16])  *$P_K$  is non-expansive if and only if  $K \subset M$  is geodesic convex.*

Finally, we recall that on a Hadamard manifold  $(M, g)$ , if  $h(p) = d_g^2(p, p_0)$ ,  $p_0 \in M$  is fixed, then

$$\text{grad}h(p) = -2 \exp_p^{-1}(p_0). \tag{5}$$

**2.3. Non-smooth calculus on manifolds.** We first recall some basic notions and results from the subdifferential calculus on Riemannian manifolds, developed by Azagra, Ferrera and López-Mesas [1], Ledyayev and Zhu [23]. Simultaneously, we introduce two subdifferential notions based on the cut locus, and we establish an analytical characterization of the limiting/Fréchet normal cone on Riemannian manifolds (see Corollary 1) which plays a crucial role in the study of Nash-Stampacchia equilibrium points.

Let  $(M, g)$  be an  $m$ -dimensional Riemannian manifold and let  $f : M \rightarrow \mathbf{R} \cup \{+\infty\}$  be a lower semicontinuous function with  $\text{dom}(f) \neq \emptyset$ . The *Fréchet-subdifferential* of  $f$  at  $p \in \text{dom}(f)$  is the set

$$\partial_F f(p) = \{dh(p) : h \in C^1(M) \text{ and } f - h \text{ attains a local minimum at } p\}.$$

**Proposition 2.2** [1, Theorem 4.3] *Let  $(M, g)$  be an  $m$ -dimensional Riemannian manifold and let  $f : M \rightarrow \mathbf{R} \cup \{+\infty\}$  be a lower semicontinuous function,  $p \in \text{dom}(f) \neq \emptyset$  and  $\xi \in T_p^*M$ . The following statements are equivalent:*

- (i)  $\xi \in \partial_F f(p)$ ;
- (ii) *For every chart  $\psi : U_p \subset M \rightarrow \mathbf{R}^m$  with  $p \in U_p$ , if  $\zeta = \xi \circ d\psi^{-1}(\psi(p))$ , we have that*

$$\liminf_{v \rightarrow 0} \frac{(f \circ \psi^{-1})(\psi(p) + v) - f(p) - \langle \zeta, v \rangle_g}{\|v\|} \geq 0;$$

- (iii) *There exists a chart  $\psi : U_p \subset M \rightarrow \mathbf{R}^m$  with  $p \in U_p$ , if  $\zeta = \xi \circ d\psi^{-1}(\psi(p))$ , then*

$$\liminf_{v \rightarrow 0} \frac{(f \circ \psi^{-1})(\psi(p) + v) - f(p) - \langle \zeta, v \rangle_g}{\|v\|} \geq 0.$$

*In addition, if  $f$  is locally bounded from below, i.e., for every  $q \in M$  there exists a neighborhood  $U_q$  of  $q$  such that  $f$  is bounded from below on  $U_q$ , the above conditions are also equivalent to*

- (iv) *There exists a function  $h \in C^1(M)$  such that  $f - h$  attains a global minimum at  $p$  and  $\xi = dh(p)$ .*

The *limiting subdifferential* and *singular subdifferential* of  $f$  at  $p \in M$  are the sets

$$\partial_L f(p) = \{\lim_k \xi_k : \xi_k \in \partial_F f(p_k), (p_k, f(p_k)) \rightarrow (p, f(p))\}$$

and

$$\partial_\infty f(p) = \{\lim_k t_k \xi_k : \xi_k \in \partial_F f(p_k), (p_k, f(p_k)) \rightarrow (p, f(p)), t_k \rightarrow 0^+\}.$$

**Proposition 2.3** [23] *Let  $(M, g)$  be a finite-dimensional Riemannian manifold and let  $f : M \rightarrow \mathbf{R} \cup \{+\infty\}$  be a lower semicontinuous function. Then, we have*

- (i)  $\partial_F f(p) \subset \partial_L f(p)$ ,  $p \in \text{dom}(f)$ ;
- (ii)  $0 \in \partial_\infty f(p)$ ,  $p \in M$ ;
- (iii) *If  $p \in \text{dom}(f)$  is a local minimum of  $f$ , then  $0 \in \partial_F f(p) \subset \partial_L f(p)$ .*

**Proposition 2.4** [23, Theorem 4.8 (Mean Value inequality)] *Let  $f : M \rightarrow \mathbf{R}$  be a continuous function bounded from below, let  $V$  be a  $C^\infty$  vector field on  $M$  and let  $c : [0, 1] \rightarrow M$  be a curve such that  $\dot{c}(t) = V(c(t))$ ,  $t \in [0, 1]$ . Then for any  $r < f(c(1)) - f(c(0))$ , any  $\varepsilon > 0$  and any open neighborhood  $U$  of  $c([0, 1])$ , there exists  $m \in U$ ,  $\xi \in \partial_F f(m)$  such that  $r < \langle \xi, V(m) \rangle_g$ .*

**Proposition 2.5** [23, Theorem 4.13 (Sum rule)] *Let  $(M, g)$  be an  $m$ -dimensional Riemannian manifold and let  $f_1, \dots, f_H : M \rightarrow \mathbf{R} \cup \{+\infty\}$  be lower semicontinuous functions. Then, for every  $p \in M$  we have either  $\partial_L(\sum_{l=1}^H f_l)(p) \subset \sum_{l=1}^H \partial_L f_l(p)$ , or there exist  $\xi_l^\infty \in \partial_\infty f_l(p)$ ,  $l = 1, \dots, H$ , not all zero such that  $\sum_{l=1}^H \xi_l^\infty = 0$ .*

The *cut-locus subdifferential* of  $f$  at  $p \in \text{dom}(f)$  is defined as

$$\partial_{cl} f(p) = \{\xi \in T_p^* M : f(q) - f(p) \geq \langle \xi, \exp_p^{-1}(q) \rangle_g \text{ for all } q \in M \setminus C_p\},$$

where  $C_p$  is the cut locus of the point  $p \in M$ . Note that  $M \setminus C_p$  is the maximal open set in  $M$  such that every element from it can be joined to  $p$  by exactly one minimizing geodesic, see Klingenberg [19, Theorem 2.1.14]. Therefore, the cut-locus subdifferential is well-defined, i.e.,  $\exp_p^{-1}(q)$  makes sense and is unique for every  $q \in M \setminus C_p$ . We first prove

**Theorem 2.1** *Let  $(M, g)$  be a Riemannian manifold and  $f : M \rightarrow \mathbf{R} \cup \{+\infty\}$  be a proper, lower semicontinuous function. Then, for every  $p \in \text{dom}(f)$  we have*

$$\partial_{cl} f(p) \subset \partial_F f(p) \subset \partial_L f(p).$$

Moreover, if  $f$  is convex, the above inclusions become equalities.

*Proof.* The last inclusion is standard, see Proposition 2.3(i). Now, let  $\xi \in \partial_{cl} f(p)$ , i.e.,  $f(q) - f(p) \geq \langle \xi, \exp_p^{-1}(q) \rangle_g$  for all  $q \in M \setminus C_p$ . In particular, the latter inequality is valid for every  $q \in B_g(p, r)$  for  $r > 0$  small enough, since  $B_g(p, r) \subset M \setminus C_p$  (for instance, when  $B_g(p, r) \subset M$  is a totally normal ball around  $p$ ). Now, by choosing  $\psi = \exp_p^{-1} : B_g(p, r) \rightarrow T_p M$  in Proposition 2.2(ii), one has that  $f(\exp_p v) - f(p) \geq \langle \xi, v \rangle_g$  for all  $v \in T_p M$ ,  $\|v\| < r$ , which implies  $\xi \in \partial_F f(p)$ .

Now, we assume in addition that  $f$  is convex, and let  $\xi \in \partial_L f(p)$ . We are going to prove that  $\xi \in \partial_{cl} f(p)$ . Since  $\xi \in \partial_L f(p)$ , we have that  $\xi = \lim_k \xi_k$  where  $\xi_k \in \partial_F f(p_k)$ ,  $(p_k, f(p_k)) \rightarrow (p, f(p))$ . By Proposition 2.2(ii), for  $\psi_k = \exp_{p_k}^{-1} : \tilde{U}_{p_k} \rightarrow T_{p_k} M$  where  $\tilde{U}_{p_k} \subset M$  is a totally normal ball centered at  $p_k$ , one has that

$$\liminf_{v \rightarrow 0} \frac{f(\exp_{p_k} v) - f(p_k) - \langle \xi_k, v \rangle_g}{\|v\|} \geq 0. \quad (6)$$

Now, fix  $q \in M \setminus C_p$ . The latter fact is equivalent to  $p \in M \setminus C_q$ , see Klingenberg [19, Lemma 2.1.11]. Since  $M \setminus C_q$  is open and  $p_k \rightarrow p$ , we may assume that  $p_k \in M \setminus C_q$ , i.e.,  $q$  and every point  $p_k$  is joined by a unique minimizing geodesic. Therefore,  $V_k = \exp_{p_k}^{-1}(q)$  is well-defined. Now, let  $\gamma_k(t) = \exp_{p_k}(tV_k)$  be the geodesic which joins  $p_k$  and  $q$ . Then (6) implies that

$$\liminf_{t \rightarrow 0^+} \frac{f(\gamma_k(t)) - f(p_k) - \langle \xi_k, tV_k \rangle_g}{\|tV_k\|} \geq 0. \quad (7)$$

Since  $f$  is convex, one has that  $f(\gamma_k(t)) \leq tf(\gamma_k(1)) + (1-t)f(\gamma_k(0))$ ,  $t \in [0, 1]$ , thus, the

latter relations imply that

$$\frac{f(q) - f(p_k) - \langle \xi_k, \exp_{p_k}^{-1}(q) \rangle_g}{d_g(p_k, q)} \geq 0.$$

Since  $f(p_k) \rightarrow f(p)$  and  $\xi = \lim_k \xi_k$ , it yields precisely that

$$f(q) - f(p) - \langle \xi, \exp_p^{-1}(q) \rangle_g \geq 0,$$

i.e.,  $\xi \in \partial_{cl} f(p)$ , which concludes the proof.  $\diamond$

**Remark 2.2** If  $(M, g)$  is a Hadamard manifold, then  $C_p = \emptyset$  for every  $p \in M$ ; in this case, the cut-locus subdifferential agrees formally with the convex subdifferential in the Euclidean setting.

Let  $K \subset M$  be a closed set. Following Ledyaev and Zhu [23], the *Fréchet-normal cone* and *limiting normal cone* of  $K$  at  $p \in K$  are the sets

$$N_F(p; K) = \partial_F \delta_K(p) \quad \text{and} \quad N_L(p; K) = \partial_L \delta_K(p),$$

where  $\delta_K$  is the indicator function of the set  $K$ , i.e.,  $\delta_K(q) = 0$  if  $q \in K$  and  $\delta_K(q) = +\infty$  if  $q \notin K$ .

The following result - which is one of our key tools to study Nash-Stampacchia equilibrium points on Riemannian manifolds - it is known for Hadamard manifolds only, see Li, López and Martín-Márquez [24] and it is a simple consequence of the above theorem.

**Corollary 1** *Let  $(M, g)$  be a Riemannian manifold,  $K \subset M$  be a closed, geodesic convex set, and  $p \in K$ . Then, we have*

$$N_F(p; K) = N_L(p; K) = \partial_{cl} \delta_K(p) = \{\xi \in T_p^* M : \langle \xi, \exp_p^{-1}(q) \rangle_g \leq 0 \text{ for all } q \in K\}.$$

*Proof.* Applying Theorem 2.1 to the indicator function  $f = \delta_K$ , we have that  $N_F(p; K) = N_L(p; K) = \partial_{cl} \delta_K(p)$ . It remains to compute the latter set explicitly. Since  $K \subset M \setminus C_p$  (note that the geodesic convexity of  $K$  assumes itself that every two points of  $K$  can be joined by a unique geodesic, thus  $K \cap C_p = \emptyset$ ) and  $\delta_K(p) = 0$ ,  $\delta_K(q) = +\infty$  for  $q \notin K$ , one has that

$$\begin{aligned} \xi \in \partial_{cl} \delta_K(p) &\Leftrightarrow \delta_K(q) - \delta_K(p) \geq \langle \xi, \exp_p^{-1}(q) \rangle_g \text{ for all } q \in M \setminus C_p \\ &\Leftrightarrow 0 \geq \langle \xi, \exp_p^{-1}(q) \rangle_g \text{ for all } q \in K, \end{aligned}$$

which ends the proof.  $\diamond$

Let  $U \subset M$  be an open subset of the Riemannian manifold  $(M, g)$ . We say that a function  $f : U \rightarrow \mathbf{R}$  is *locally Lipschitz at  $p \in U$*  if there exist an open neighborhood  $U_p \subset U$  of  $p$

and a number  $C_p > 0$  such that for every  $q_1, q_2 \in U_p$ ,

$$|f(q_1) - f(q_2)| \leq C_p d_g(q_1, q_2).$$

The function  $f : U \rightarrow \mathbf{R}$  is *locally Lipschitz* on  $(U, g)$  if it is locally Lipschitz at every  $p \in U$ .

Fix  $p \in U$ ,  $v \in T_p M$ , and let  $\tilde{U}_p \subset U$  be a totally normal neighborhood of  $p$ . If  $q \in \tilde{U}_p$ , following [1, Section 5], for small values of  $|t|$ , we may introduce

$$\sigma_{q,v}(t) = \exp_q(tw), \quad w = d(\exp_q^{-1} \circ \exp_p)_{\exp_p^{-1}(q)} v.$$

If the function  $f : U \rightarrow \mathbf{R}$  is locally Lipschitz on  $(U, g)$ , then

$$f^0(p; v) = \limsup_{q \rightarrow p, t \rightarrow 0^+} \frac{f(\sigma_{q,v}(t)) - f(q)}{t}$$

is called the *Clarke generalized derivative of  $f$  at  $p \in U$  in direction  $v \in T_p M$* , and

$$\partial_C f(p) = \text{co}(\partial_L f(p))$$

is the *Clarke subdifferential of  $f$  at  $p \in U$* , where 'co' stands for the convex hull. When  $f : U \rightarrow \mathbf{R}$  is a  $C^1$  functional at  $p \in U$  then

$$\partial_C f(p) = \partial_L f(p) = \partial_F f(p) = \{df(p)\}, \quad (8)$$

see [1, Proposition 4.6]. Moreover, when  $(M, g)$  is the standard Euclidean space, the Clarke subdifferential and the Clarke generalized gradient agree, see Clarke [11].

One can easily prove that the function  $f^0(\cdot; \cdot)$  is upper-semicontinuous on  $TU = \cup_{p \in U} T_p M$  and  $f^0(p; \cdot)$  is positive homogeneous and subadditive on  $T_p M$ , thus convex. In addition, if  $U \subset M$  is geodesic convex and  $f : U \rightarrow \mathbf{R}$  is convex, then

$$f^0(p; v) = \lim_{t \rightarrow 0^+} \frac{f(\exp_p(tv)) - f(p)}{t}, \quad (9)$$

see Claim 5.4 and the first relation on p. 341 of [1].

**Proposition 2.6** [23, Corollary 5.3] *Let  $(M, g)$  be a Riemannian manifold and let  $f : M \rightarrow \mathbf{R} \cup \{+\infty\}$  be a lower semicontinuous function. Then the following statements are equivalent:*

- (i)  $f$  is locally Lipschitz at  $p \in M$ ;
- (ii)  $\partial_C f$  is bounded in a neighborhood of  $p \in M$ ;
- (iii)  $\partial_\infty f(p) = \{0\}$ .

**Proposition 2.7** *Let  $f, g : M \rightarrow \mathbf{R} \cup \{+\infty\}$  be two proper, lower semicontinuous functions. Then, for every  $p \in \text{dom}(f) \cap \text{dom}(g)$  with  $\partial_{cl}f(p) \neq \emptyset \neq \partial_{cl}g(p)$  we have  $\partial_{cl}f(p) + \partial_{cl}g(p) \subset \partial_{cl}(f+g)(p)$ . Moreover, if both functions are convex and  $f$  is locally bounded, the inclusion is equality.*

*Proof.* The first part is trivial. For the second part,  $f$  is a locally Lipschitz function (see Azagra, Ferrera, and López-Mesas [1, Proposition 5.2]), thus Theorem 1 and Propositions 2.5 & 2.6 give  $\partial_{cl}(f+g)(p) \subset \partial_L(f+g)(p) \subset \partial_Lf(p) + \partial_Lg(p) = \partial_{cl}f(p) + \partial_{cl}g(p)$ .  $\diamond$

Let  $f : U \rightarrow \mathbf{R}$  be a locally Lipschitz function and  $p \in U$ . We consider the *Clarke 0-subdifferential* of  $f$  at  $p$  as

$$\begin{aligned} \partial_0f(p) &= \{\xi \in T_p^*M : f^0(p; \exp_p^{-1}(q)) \geq \langle \xi, \exp_p^{-1}(q) \rangle_g \text{ for all } q \in U \setminus C_p\} \\ &= \{\xi \in T_p^*M : f^0(p; v) \geq \langle \xi, v \rangle_g \text{ for all } v \in T_pM\}. \end{aligned}$$

**Theorem 2.2** *Let  $(M, g)$  be a Riemannian manifold,  $U \subset M$  be open,  $f : U \rightarrow \mathbf{R}$  be a locally Lipschitz function, and  $p \in U$ . Then,*

$$\partial_0f(p) = \partial_{cl}(f^0(p; \exp_p^{-1}(\cdot)))(p) = \partial_L(f^0(p; \exp_p^{-1}(\cdot)))(p) = \partial_Cf(p).$$

*Proof. Step 1.*  $\partial_0f(p) = \partial_{cl}(f^0(p; \exp_p^{-1}(\cdot)))(p)$ . It follows from the definitions.

**Step 2.**  $\partial_0f(p) = \partial_L(f^0(p; \exp_p^{-1}(\cdot)))(p)$ .

The inclusion " $\subset$ " follows from Step 1 and Theorem 2.1. For the converse, we notice that  $f^0(p; \exp_p^{-1}(\cdot))$  is locally Lipschitz in a neighborhood of  $p$ ; indeed,  $f^0(p; \cdot)$  is convex on  $T_pM$  and  $\exp_p$  is a local diffeomorphism on a neighborhood of the origin of  $T_pM$ . Now, let  $\xi \in \partial_L(f^0(p; \exp_p^{-1}(\cdot)))(p)$ . Then,  $\xi = \lim_k \xi_k$  where  $\xi_k \in \partial_F(f^0(p; \exp_p^{-1}(\cdot)))(p_k)$ ,  $p_k \rightarrow p$ . By Proposition 2.2(ii), for  $\psi = \exp_p^{-1} : \tilde{U}_p \rightarrow T_pM$  where  $\tilde{U}_p \subset M$  is a totally normal ball centered at  $p$ , one has that

$$\liminf_{v \rightarrow 0} \frac{f^0(p; \exp_p^{-1}(p_k) + v) - f^0(p; \exp_p^{-1}(p_k)) - \langle \xi_k((d \exp_p)(\exp_p^{-1}(p_k))), v \rangle_g}{\|v\|} \geq 0. \quad (10)$$

In particular, if  $q \in M \setminus C_p$  is fixed arbitrarily and  $v = t \exp_p^{-1}(q)$  for  $t > 0$  small, the convexity of  $f^0(p; \cdot)$  and relation (10) yield that

$$f^0(p; \exp_p^{-1}(q)) \geq \langle \xi_k((d \exp_p)(\exp_p^{-1}(p_k))), \exp_p^{-1}(q) \rangle_g.$$

Since  $\xi = \lim_k \xi_k$ ,  $p_k \rightarrow p$  and  $d(\exp_p)(0) = \text{id}_{T_pM}$  (see (3)), we obtain that

$$f^0(p; \exp_p^{-1}(q)) \geq \langle \xi, \exp_p^{-1}(q) \rangle_g,$$

i.e.,  $\xi \in \partial_0f(p)$ . This concludes Step 2.

**Step 3.**  $\partial_0 f(p) = \partial_C f(p)$ .

First, we prove the inclusion  $\partial_0 f(p) \subset \partial_C f(p)$ . Here, we follow Borwein and Zhu [6, Theorem 5.2.16], see also Clarke, Ledyaev, Stern and Wolenski [12, Theorem 6.1]. Let  $v \in T_p M$  be fixed arbitrarily. The definition of  $f^0(p; v)$  shows that one can choose  $t_k \rightarrow 0^+$  and  $q_k \rightarrow p$  such that

$$f^0(p; v) = \lim_{k \rightarrow \infty} \frac{f(\sigma_{q_k, v}(t_k)) - f(q_k)}{t_k}.$$

Fix  $\varepsilon > 0$ . For large  $k \in \mathbb{N}$ , let  $c_k : [0, 1] \rightarrow M$  be the unique geodesic joining the points  $q_k$  and  $\sigma_{q_k, v}(t_k)$ , i.e.,  $c_k(t) = \exp_{q_k}(t \exp_{q_k}^{-1}(\sigma_{q_k, v}(t_k)))$  and let  $U_k = \cup_{t \in [0, 1]} B_g(c_k(t), \varepsilon t_k)$  its  $(\varepsilon t_k)$ -neighborhood. Consider also a  $C^\infty$  vector field  $V$  on  $U_k$  such that  $\dot{c}_k(t) = V(c_k(t))$ ,  $t \in [0, 1]$ . Now, applying Proposition 2.4 with  $r_k = f(c_k(1)) - f(c_k(0)) - \varepsilon t_k$ , one can find  $m_k = m_k(t_k, q_k, v) \in U_k$  and  $\xi_k \in \partial_F f(m_k)$  such that  $r_k < \langle \xi_k, V(m_k) \rangle_g$ . The latter inequality is equivalent to

$$\frac{f(\sigma_{q_k, v}(t_k)) - f(q_k)}{t_k} < \varepsilon + \langle \xi_k, V(m_k)/t_k \rangle_g.$$

Since  $f$  is locally Lipschitz,  $\partial_F f$  is bounded in a neighborhood of  $p$ , see Proposition 2.6, thus the sequence  $\{\xi_k\}$  is bounded on  $TM$ . We can choose a convergent subsequence (still denoting by  $\{\xi_k\}$ ), and let  $\xi_L = \lim_k \xi_k$ . From construction,  $\xi_L \in \partial_L f(p) \subset \partial_C f(p)$ . Since  $m_k \rightarrow p$ , according to (3), we have that  $\lim_{k \rightarrow \infty} V(m_k)/t_k = v$ . Thus, letting  $k \rightarrow \infty$  in the latter inequality, the arbitrariness of  $\varepsilon > 0$  yields that

$$f^0(p; v) \leq \langle \xi_L, v \rangle_g.$$

Now, taking into account that  $f^0(p; v) = \max\{\langle \xi, v \rangle_g : \xi \in \partial_0 f(p)\}$ , we obtain that

$$\max\{\langle \xi, v \rangle_g : \xi \in \partial_0 f(p)\} = f^0(p; v) \leq \langle \xi_L, v \rangle_g \leq \sup\{\langle \xi, v \rangle_g : \xi \in \partial_C f(p)\}.$$

Hörmander's result (see [12]) shows that this inequality in terms of support functions of convex sets is equivalent to the inclusion  $\partial_0 f(p) \subset \partial_C f(p)$ .

For the converse, it is enough to prove that  $\partial_L f(p) \subset \partial_0 f(p)$  since the latter set is convex. Let  $\xi \in \partial_L f(p)$ . Then, we have  $\xi = \lim_k \xi_k$  where  $\xi_k \in \partial_F f(p_k)$  and  $p_k \rightarrow p$ . A similar argument as in the proof of Theorem 2.1 (see relation (7)) gives that for every  $q \in M \setminus C_p$  and  $k \in \mathbb{N}$ , we have

$$\liminf_{t \rightarrow 0^+} \frac{f(\exp_{p_k}(t \exp_{p_k}^{-1}(q))) - f(p_k) - \langle \xi_k, t \exp_{p_k}^{-1}(q) \rangle_g}{\|t \exp_{p_k}^{-1}(q)\|} \geq 0.$$

Since  $\|\exp_{p_k}^{-1}(q)\| = d_g(p_k, q) \geq c_0 > 0$ , by the definition of the Clarke generalized derivative  $f^0$  and the above inequality, one has that

$$f^0(p_k; \exp_{p_k}^{-1}(q)) \geq \langle \xi_k, \exp_{p_k}^{-1}(q) \rangle_g.$$

The upper semicontinuity of  $f^0(\cdot; \cdot)$  and the fact that  $\xi = \lim_k \xi_k$  imply that

$$f^0(p; \exp_p^{-1}(q)) \geq \limsup_k f^0(p_k; \exp_{p_k}^{-1}(q)) \geq \limsup_k \langle \xi_k, \exp_{p_k}^{-1}(q) \rangle_g = \langle \xi, \exp_p^{-1}(q) \rangle_g,$$

i.e.,  $\xi \in \partial_0 f(p)$ , which concludes the proof of Step 3.  $\diamond$

**2.4. Dynamical systems on manifolds.** In this subsection we recall the existence of a local solution for a Cauchy-type problem defined on Riemannian manifolds and its viability relative to a closed set.

Let  $(M, g)$  be a finite-dimensional Riemannian manifold and  $G : M \rightarrow TM$  be a vector field on  $M$ , i.e.,  $G(p) \in T_p M$  for every  $p \in M$ . We assume in the sequel that  $G : M \rightarrow TM$  is a  $C^{1-0}$  vector field (i.e., locally Lipschitz); then the dynamical system

$$(DS)_G \quad \begin{cases} \dot{\eta}(t) = G(\eta(t)), \\ \eta(0) = p_0, \end{cases}$$

has a unique maximal semiflow  $\eta : [0, T) \rightarrow M$ , see Chang [9, p. 15]. In particular,  $\eta$  is an absolutely continuous function such that  $[0, T) \ni t \mapsto \dot{\eta}(t) \in T_{\eta(t)} M$  and it verifies  $(DS)_G$  for a.e.  $t \in [0, T)$ .

A set  $K \subset M$  is *invariant with respect to the solutions of  $(DS)_G$*  if for every initial point  $p_0 \in K$  the unique maximal semiflow/orbit  $\eta : [0, T) \rightarrow M$  of  $(DS)_G$  fulfills the property that  $\eta(t) \in K$  for every  $t \in [0, T)$ . We introduce the Hamiltonian function as

$$H_G(p, \xi) = \langle \xi, G(p) \rangle_g, \quad (p, \xi) \in M \times T_p^* M.$$

Note that  $H_G(p, dh(p)) < \infty$  for every  $p \in M$  and  $h \in C^1(M)$ . Therefore, after a suitable adaptation of the results from Ledyaev and Zhu [23, Subsection 6.2] we may state

**Proposition 2.8** *Let  $G : M \rightarrow TM$  be a  $C^{1-0}$  vector field and  $K \subset M$  be a non-empty closed set. The following statements are equivalent:*

- (i)  *$K$  is invariant with respect to the solutions of  $(DS)_G$ ;*
- (ii)  *$H_G(p, \xi) \leq 0$  for any  $p \in K$  and  $\xi \in N_F(p; K)$ .*

### 3 Comparison of Nash-type equilibria

Let  $K_1, \dots, K_n$  ( $n \geq 2$ ) be non-empty sets, corresponding to the strategies of  $n$  players and  $f_i : K_1 \times \dots \times K_n \rightarrow \mathbf{R}$  ( $i \in \{1, \dots, n\}$ ) be the payoff functions, respectively. Throughout the paper, the following notations/conventions are used:

- $\mathbf{K} = K_1 \times \dots \times K_n$ ;  $\mathbf{f} = (f_1, \dots, f_n)$ ;  $(\mathbf{f}, \mathbf{K}) = (f_1, \dots, f_n; K_1, \dots, K_n)$ ;
- $\mathbf{p} = (p_1, \dots, p_n)$ ;
- $\mathbf{p}_{-i}$  is a strategy profile of all players except for player  $i$ ;  
 $(q_i, \mathbf{p}_{-i}) = (p_1, \dots, p_{i-1}, q_i, p_{i+1}, \dots, p_n)$ ; in particular,  $(p_i, \mathbf{p}_{-i}) = \mathbf{p}$ ;



- $\mathbf{K}_{-i}$  is the strategy set profile of all players except for player  $i$ ;  
 $(U_i, \mathbf{K}_{-i}) = K_1 \times \dots \times K_{i-1} \times U_i \times K_{i+1} \times \dots \times K_n$  for some  $U_i \supset K_i$ .

**Definition 3.1** *The set of Nash equilibrium points for  $(\mathbf{f}, \mathbf{K})$  is*

$$\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \{\mathbf{p} \in \mathbf{K} : f_i(q_i, \mathbf{p}_{-i}) \geq f_i(\mathbf{p}) \text{ for all } q_i \in K_i, i \in \{1, \dots, n\}\}.$$

The main result of the paper [20] states that in a quite general framework the set of Nash equilibrium points for  $(\mathbf{f}, \mathbf{K})$  is not empty. More precisely, we have

**Proposition 3.1** [20] *Let  $(M_i, g_i)$  be finite-dimensional Riemannian manifolds;  $K_i \subset M_i$  be non-empty, compact, geodesic convex sets; and  $f_i : \mathbf{K} \rightarrow \mathbf{R}$  be continuous functions such that  $K_i \ni q_i \mapsto f_i(q_i, \mathbf{p}_{-i})$  is convex on  $K_i$  for every  $\mathbf{p}_{-i} \in \mathbf{K}_{-i}$ ,  $i \in \{1, \dots, n\}$ . Then there exists at least one Nash equilibrium point for  $(\mathbf{f}, \mathbf{K})$ , i.e.,  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \neq \emptyset$ .*

Similarly to Proposition 3.1, let us assume that for every  $i \in \{1, \dots, n\}$ , one can find a finite-dimensional Riemannian manifold  $(M_i, g_i)$  such that the strategy set  $K_i$  is closed and geodesic convex in  $(M_i, g_i)$ . Let  $\mathbf{M} = M_1 \times \dots \times M_n$  be the product manifold with its standard Riemannian product metric

$$\mathbf{g}(\mathbf{V}, \mathbf{W}) = \sum_{i=1}^n g_i(V_i, W_i) \quad (11)$$

for every  $\mathbf{V} = (V_1, \dots, V_n)$ ,  $\mathbf{W} = (W_1, \dots, W_n) \in T_{p_1}M_1 \times \dots \times T_{p_n}M_n = T_{\mathbf{p}}\mathbf{M}$ . Let  $\mathbf{U} = U_1 \times \dots \times U_n \subset \mathbf{M}$  be an open set such that  $\mathbf{K} \subset \mathbf{U}$ ; we always mean that  $U_i$  inherits the Riemannian structure of  $(M_i, g_i)$ . Let

$$\begin{aligned} \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})} = \{ \mathbf{f} = (f_1, \dots, f_n) \in C^0(\mathbf{K}, \mathbf{R}^n) : & f_i : (U_i, \mathbf{K}_{-i}) \rightarrow \mathbf{R} \text{ is continuous and} \\ & f_i(\cdot, \mathbf{p}_{-i}) \text{ is locally Lipschitz on } (U_i, g_i) \\ & \text{for all } \mathbf{p}_{-i} \in \mathbf{K}_{-i}, i \in \{1, \dots, n\} \}. \end{aligned}$$

The next notion has been introduced in [20].

**Definition 3.2** *Let  $\mathbf{f} \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ . The set of Nash-Clarke equilibrium points for  $(\mathbf{f}, \mathbf{K})$  is*

$$\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) = \{\mathbf{p} \in \mathbf{K} : f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i)) \geq 0 \text{ for all } q_i \in K_i, i \in \{1, \dots, n\}\}.$$

Here,  $f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i))$  denotes the Clarke generalized derivative of  $f_i(\cdot, \mathbf{p}_{-i})$  at point  $p_i \in K_i$  in direction  $\exp_{p_i}^{-1}(q_i) \in T_{p_i}M_i$ . More precisely,

$$f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i)) = \limsup_{q \rightarrow p_i, q \in U_i, t \rightarrow 0^+} \frac{f_i(\sigma_{q, \exp_{p_i}^{-1}(q_i)}(t), \mathbf{p}_{-i}) - f_i(q, \mathbf{p}_{-i})}{t}, \quad (12)$$

where  $\sigma_{q,v}(t) = \exp_q(tw)$ , and  $w = d(\exp_q^{-1} \circ \exp_{p_i})_{\exp_{p_i}^{-1}(q)} v$  for  $v \in T_{p_i}M_i$ , and  $t > 0$  is small enough. By exploiting a minimax result of McClendon [27], the following existence result is available concerning the Nash-Clarke points for  $(\mathbf{f}, \mathbf{K})$ .

**Proposition 3.2** [20] *Let  $(M_i, g_i)$  be complete finite-dimensional Riemannian manifolds;  $K_i \subset M_i$  be non-empty, compact, geodesic convex sets; and  $\mathbf{f} \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$  such that for every  $\mathbf{p} \in \mathbf{K}$ ,  $i \in \{1, \dots, n\}$ ,  $K_i \ni q_i \mapsto f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i))$  is convex and  $f_i^0$  is upper semicontinuous on its domain of definition. Then  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) \neq \emptyset$ .*

**Remark 3.1** Although Proposition 3.2 gives a possible approach to locate Nash equilibria on Riemannian manifolds, its applicability is quite reduced. Indeed,  $f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(\cdot))$  has no convexity property in general, unless we are in the Euclidean setting or the set  $K_i$  is a geodesic segment, see [20]. For instance, if  $\mathbf{H}^2$  is the standard Poincaré upper-plane with the metric  $g_{\mathbf{H}} = (\frac{\delta_{ij}}{y^2})$  and we consider the function  $f : \mathbf{H}^2 \times \mathbf{R} \rightarrow \mathbf{R}$ ,  $f((x, y), r) = rx$  and the geodesic segment  $\gamma(t) = (1, e^t)$  in  $\mathbf{H}^2$ ,  $t \in [0, 1]$ , the function

$$t \mapsto f_1^0(((2, 1), r); \exp_{(2,1)}^{-1}(\gamma(t))) = r \left( e^{2t} \frac{\sinh 2}{2} + e^t \cosh 1 \sqrt{e^{2t}(\cosh 1)^2 - 1} \right)^{-1}$$

is not convex.

The limited applicability of Proposition 3.2 comes from the involved form of the set  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$  which motivates the introduction and study of the following concept which plays the central role in the present paper.

**Definition 3.3** *Let  $\mathbf{f} \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ . The set of Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  is*

$$\begin{aligned} \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{ \mathbf{p} \in \mathbf{K} : & \exists \xi_C^i \in \partial_C^i f_i(\mathbf{p}) \text{ such that } \langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0, \\ & \text{for all } q_i \in K_i, i \in \{1, \dots, n\} \}. \end{aligned}$$

Here,  $\partial_C^i f_i(\mathbf{p})$  denotes the Clarke subdifferential of the function  $f_i(\cdot, \mathbf{p}_{-i})$  at point  $p_i \in K_i$ , i.e.,  $\partial_C f_i(\cdot, \mathbf{p}_{-i})(p_i) = \text{co}(\partial_L f_i(\cdot, \mathbf{p}_{-i})(p_i))$ .

Our first aim is to compare the three Nash-type points introduced in Definitions 3.1-3.3. Before to do that, we introduce another two classes of functions. If  $U_i \subset M_i$  is geodesic convex for every  $i \in \{1, \dots, n\}$ , we may define

$$\begin{aligned} \mathcal{K}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})} = \{ \mathbf{f} \in C^0(\mathbf{K}, \mathbf{R}^n) : & f_i : (U_i, \mathbf{K}_{-i}) \rightarrow \mathbf{R} \text{ is continuous and } f_i(\cdot, \mathbf{p}_{-i}) \text{ is} \\ & \text{convex on } (U_i, g_i) \text{ for all } \mathbf{p}_{-i} \in \mathbf{K}_{-i}, i \in \{1, \dots, n\} \}, \end{aligned}$$

and

$$\mathcal{C}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})} = \{\mathbf{f} \in C^0(\mathbf{K}, \mathbf{R}^n) : f_i : (U_i, \mathbf{K}_{-i}) \rightarrow \mathbf{R} \text{ is continuous and } f_i(\cdot, \mathbf{p}_{-i}) \text{ is of class } C^1 \text{ on } (U_i, g_i) \text{ for all } \mathbf{p}_{-i} \in \mathbf{K}_{-i}, i \in \{1, \dots, n\}\}.$$

**Remark 3.2** Due to Azagra, Ferrera and López-Mesas [1, Proposition 5.2], one has that  $\mathcal{K}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})} \subset \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ . Moreover, it is clear that  $\mathcal{C}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})} \subset \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ .

The main result of this section reads as follows.

**Theorem 3.1** *Let  $(M_i, g_i)$  be finite-dimensional Riemannian manifolds;  $K_i \subset M_i$  be non-empty, closed, geodesic convex sets;  $U_i \subset M_i$  be open sets containing  $K_i$ ; and  $f_i : \mathbf{K} \rightarrow \mathbf{R}$  be some functions,  $i \in \{1, \dots, n\}$ . Then, we have*

- (i)  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$  whenever  $\mathbf{f} \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ ;
- (ii)  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$  whenever  $\mathbf{f} \in \mathcal{K}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ ;

*Proof.* (i) First, we prove that  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ . Indeed, we have  $\mathbf{p} \in \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) \Leftrightarrow$

$$\begin{aligned} &\Leftrightarrow f_i(q_i, \mathbf{p}_{-i}) \geq f_i(\mathbf{p}) \text{ for all } q_i \in K_i, i \in \{1, \dots, n\} \\ &\Leftrightarrow 0 \in \partial_{cl}(f_i(\cdot, \mathbf{p}_{-i}) + \delta_{K_i})(p_i), i \in \{1, \dots, n\} \\ &\Rightarrow 0 \in \partial_L(f_i(\cdot, \mathbf{p}_{-i}) + \delta_{K_i})(p_i), i \in \{1, \dots, n\} \quad (\text{cf. Theorem 2.1}) \\ &\Rightarrow 0 \in \partial_L f_i(\cdot, \mathbf{p}_{-i})(p_i) + \partial_L \delta_{K_i}(p_i), i \in \{1, \dots, n\} \quad (\text{cf. Propositions 2.5 \& 2.6}) \\ &\Rightarrow 0 \in \partial_C f_i(\cdot, \mathbf{p}_{-i})(p_i) + \partial_L \delta_{K_i}(p_i), i \in \{1, \dots, n\} \\ &\Leftrightarrow 0 \in \partial_C^i f_i(\mathbf{p}) + N_L(p_i; K_i), i \in \{1, \dots, n\} \\ &\Leftrightarrow \exists \xi_C^i \in \partial_C^i f_i(\mathbf{p}) \text{ such that } \langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i} \geq 0 \text{ for all } q_i \in K_i, i \in \{1, \dots, n\} \\ &\quad (\text{cf. Corollary 1}) \\ &\Leftrightarrow \mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}). \end{aligned}$$

Now, we prove  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ ; more precisely, we have  $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \Leftrightarrow$

$$\begin{aligned} &\Leftrightarrow 0 \in \partial_C^i f_i(\mathbf{p}) + N_L(p_i; K_i), i \in \{1, \dots, n\} \\ &\Leftrightarrow 0 \in \partial_C^i f_i(\mathbf{p}) + \partial_{cl} \delta_{K_i}(p_i), i \in \{1, \dots, n\} \quad (\text{cf. Corollary 1}) \\ &\Leftrightarrow 0 \in \partial_{cl}(f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(\cdot)))(p_i) + \partial_{cl} \delta_{K_i}(p_i), i \in \{1, \dots, n\} \quad (\text{cf. Theorem 2.2}) \\ &\Rightarrow 0 \in \partial_{cl}(f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(\cdot)) + \delta_{K_i})(p_i), i \in \{1, \dots, n\} \quad (\text{cf. Proposition 2.7}) \\ &\Leftrightarrow f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i)) \geq 0 \text{ for all } q_i \in K_i, i \in \{1, \dots, n\} \\ &\Leftrightarrow \mathbf{p} \in \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}). \end{aligned}$$

In order to prove  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ , we recall that  $f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(\cdot))$  is locally Lipschitz in a neighborhood of  $p_i$ . Thus, we have  
 $\mathbf{p} \in \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) \Leftrightarrow$

$$\begin{aligned} &\Leftrightarrow 0 \in \partial_{cl}(f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(\cdot)) + \delta_{K_i})(p_i), \quad i \in \{1, \dots, n\} \\ &\Rightarrow 0 \in \partial_L(f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(\cdot)) + \delta_{K_i})(p_i), \quad i \in \{1, \dots, n\} \quad (\text{cf. Theorem 2.1}) \\ &\Rightarrow 0 \in \partial_L(f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(\cdot)))(p_i) + \partial_L \delta_{K_i}(p_i), \quad i \in \{1, \dots, n\} \quad (\text{cf. Propositions 2.5 \& 2.6}) \\ &\Leftrightarrow 0 \in \partial_C(f_i(\cdot, \mathbf{p}_{-i}))(p_i) + \partial_L \delta_{K_i}(p_i), \quad i \in \{1, \dots, n\} \quad (\text{cf. Theorem 2.2}) \\ &\Leftrightarrow 0 \in \partial_C^i(f_i(\mathbf{p})) + N_L(p_i; K_i), \quad i \in \{1, \dots, n\} \\ &\Leftrightarrow \mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}). \end{aligned}$$

(ii) Due to (i) and Remark 3.2, it is enough to prove that  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K}) \subset \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ . Let  $\mathbf{p} \in \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ , i.e., for every  $i \in \{1, \dots, n\}$  and  $q_i \in K_i$ ,

$$f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i)) \geq 0. \quad (13)$$

Fix  $i \in \{1, \dots, n\}$  and  $q_i \in K_i$  arbitrary. Since  $f_i(\cdot, \mathbf{p}_{-i})$  is convex on  $(U_i, g_i)$ , on account of (9), we have

$$f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i)) = \lim_{t \rightarrow 0^+} \frac{f_i(\exp_{p_i}(t \exp_{p_i}^{-1}(q_i)), \mathbf{p}_{-i}) - f_i(\mathbf{p})}{t}. \quad (14)$$

Note that the function

$$R(t) = \frac{f_i(\exp_{p_i}(t \exp_{p_i}^{-1}(q_i)), \mathbf{p}_{-i}) - f_i(\mathbf{p})}{t}$$

is well-defined on the whole interval  $(0, 1]$ ; indeed,  $t \mapsto \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))$  is the minimal geodesic joining the points  $p_i \in K_i$  and  $q_i \in K_i$  which belongs to  $K_i \subset U_i$ . Moreover, it is well-known that  $t \mapsto R(t)$  is non-decreasing on  $(0, 1]$ . Consequently,

$$f_i(q_i, \mathbf{p}_{-i}) - f_i(\mathbf{p}) = f_i(\exp_{p_i}(\exp_{p_i}^{-1}(q_i)), \mathbf{p}_{-i}) - f_i(\mathbf{p}) = R(1) \geq \lim_{t \rightarrow 0^+} R(t).$$

Now, (13) and (14) give that  $\lim_{t \rightarrow 0^+} R(t) \geq 0$ , which concludes the proof.  $\diamond$

**Remark 3.3** (a) As we can see, the key tool in the proof of  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$  is the locally Lipschitz property of the function  $f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(\cdot))$  near  $p_i$ .

(b) In [20] we considered the sets  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$  and  $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ . Note however that the Nash-Stampacchia concept is more appropriate to find Nash equilibrium points in general contexts, see also the applications in §6 for both compact and non-compact cases. Moreover, via  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$  we realize that the optimal geometrical framework to develop this study is the class of Hadamard manifolds. In the next sections we develop this approach.

#### 4 Nash-Stampacchia equilibria on Hadamard manifolds: characterization, existence and stability

Let  $(M_i, g_i)$  be finite-dimensional Hadamard manifolds,  $i \in \{1, \dots, n\}$ . Standard arguments show that  $(\mathbf{M}, \mathbf{g})$  is also a Hadamard manifold, see Ballmann [2, Example 4, p.147] and O'Neill [32, Lemma 40, p. 209]. Moreover, on account of the characterization of (warped) product geodesics, see O'Neill [32, Proposition 38, p. 208], if  $\exp_{\mathbf{p}}$  denotes the usual exponential map on  $(\mathbf{M}, \mathbf{g})$  at  $\mathbf{p} \in \mathbf{M}$ , then for every  $\mathbf{V} = (V_1, \dots, V_n) \in T_{\mathbf{p}}\mathbf{M}$ , we have

$$\exp_{\mathbf{p}}(\mathbf{V}) = (\exp_{p_1}(V_1), \dots, \exp_{p_n}(V_n)).$$

We consider that  $K_i \subset M_i$  are non-empty, closed, geodesic convex sets and  $U_i \subset M_i$  are open sets containing  $K_i$ ,  $i \in \{1, \dots, n\}$ .

Let  $\mathbf{f} \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ . The *diagonal Clarke subdifferential* of  $\mathbf{f} = (f_1, \dots, f_n)$  at  $\mathbf{p} \in \mathbf{K}$  is

$$\partial_C^\Delta \mathbf{f}(\mathbf{p}) = (\partial_C^1 f_1(\mathbf{p}), \dots, \partial_C^n f_n(\mathbf{p})).$$

From the definition of the metric  $\mathbf{g}$ , for every  $\mathbf{p} \in \mathbf{K}$  and  $\mathbf{q} \in \mathbf{M}$  it turns out that

$$\langle \xi_C^\Delta, \exp_{\mathbf{p}}^{-1}(\mathbf{q}) \rangle_{\mathbf{g}} = \sum_{i=1}^n \langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \rangle_{g_i}, \quad \xi_C^\Delta = (\xi_C^1, \dots, \xi_C^n) \in \partial_C^\Delta \mathbf{f}(\mathbf{p}). \quad (15)$$

**4.1. Nash-Stampacchia equilibrium points versus fixed points of  $A_\alpha^{\mathbf{f}}$ .** For each  $\alpha > 0$  and  $\mathbf{f} \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ , we define the set-valued map  $A_\alpha^{\mathbf{f}} : \mathbf{K} \rightarrow 2^{\mathbf{K}}$  by

$$A_\alpha^{\mathbf{f}}(\mathbf{p}) = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \partial_C^\Delta \mathbf{f}(\mathbf{p}))), \quad \mathbf{p} \in \mathbf{K}.$$

Note that for each  $\mathbf{p} \in \mathbf{K}$ , the set  $A_\alpha^{\mathbf{f}}(\mathbf{p})$  is non-empty and compact. The following result plays a crucial role in our further investigations.

**Theorem 4.1** *Let  $(M_i, g_i)$  be finite-dimensional Hadamard manifolds;  $K_i \subset M_i$  be non-empty, closed, geodesic convex sets;  $U_i \subset M_i$  be open sets containing  $K_i$ ,  $i \in \{1, \dots, n\}$ ; and  $\mathbf{f} \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ . Then the following statements are equivalent:*

- (i)  $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ ;
- (ii)  $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$  for all  $\alpha > 0$ ;
- (iii)  $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$  for some  $\alpha > 0$ .

*Proof.* In view of relation (15) and the identification between  $T_{\mathbf{p}}\mathbf{M}$  and  $T_{\mathbf{p}}^*\mathbf{M}$ , see (1), we have that

$$\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \Leftrightarrow \exists \xi_C^\Delta = (\xi_C^1, \dots, \xi_C^n) \in \partial_C^\Delta \mathbf{f}(\mathbf{p}) \text{ such that} \quad (16)$$

$$\begin{aligned}
& \langle \xi_C^\Delta, \exp_{\mathbf{p}}^{-1}(\mathbf{q}) \rangle_{\mathbf{g}} \geq 0 \text{ for all } \mathbf{q} \in \mathbf{K} \\
& \Leftrightarrow \exists \xi_C^\Delta = (\xi_C^1, \dots, \xi_C^n) \in \partial_C^\Delta \mathbf{f}(\mathbf{p}) \text{ such that} \\
& \mathbf{g}(-\alpha \xi_C^\Delta, \exp_{\mathbf{p}}^{-1}(\mathbf{q})) \leq 0 \text{ for all } \mathbf{q} \in \mathbf{K} \text{ and} \\
& \text{for all/some } \alpha > 0.
\end{aligned}$$

On the other hand, let  $\gamma, \sigma : [0, 1] \rightarrow \mathbf{M}$  be the unique minimal geodesics defined by  $\gamma(t) = \exp_{\mathbf{p}}(-t\alpha \xi_C^\Delta)$  and  $\sigma(t) = \exp_{\mathbf{p}}(t \exp_{\mathbf{p}}^{-1}(\mathbf{q}))$  for any fixed  $\alpha > 0$  and  $\mathbf{q} \in \mathbf{K}$ . Since  $\mathbf{K}$  is geodesic convex in  $(\mathbf{M}, \mathbf{g})$ , then  $\text{Im}\sigma \subset \mathbf{K}$  and

$$\mathbf{g}(\dot{\gamma}(0), \dot{\sigma}(0)) = \mathbf{g}(-\alpha \xi_C^\Delta, \exp_{\mathbf{p}}^{-1}(\mathbf{q})). \quad (17)$$

Taking into account relation (17) and Proposition 2.1 (i), i.e., the validity of the obtuse-angle property on the Hadamard manifold  $(\mathbf{M}, \mathbf{g})$ , (16) is equivalent to

$$\mathbf{p} = \gamma(0) = P_{\mathbf{K}}(\gamma(1)) = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha \xi_C^\Delta)),$$

which is nothing but  $\mathbf{p} \in A_\alpha^{\mathbf{f}}(\mathbf{p})$ . ◇

**Remark 4.1** Note that the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) hold for arbitrarily Riemannian manifolds, see Remark 2.1 (b). These implications are enough to find Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  via fixed points of the map  $A_\alpha^{\mathbf{f}}$ . However, in the sequel we exploit further aspects of the Hadamard manifolds as non-expansiveness of the projection operator of geodesic convex sets and a Rauch-type comparison property. Moreover, in the spirit of Nash's original idea that Nash equilibria appear exactly as fixed points of a specific map, Theorem 4.1 provides a full characterization of Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  via the fixed points of the set-valued map  $A_\alpha^{\mathbf{f}}$  when  $(M_i, g_i)$  are Hadamard manifolds.

In the sequel, two cases will be considered to guarantee Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$ , depending on the compactness of the strategy sets  $K_i$ .

**4.2. Nash-Stampacchia equilibrium points; compact case.** Our first result guarantees the existence of a Nash-Stampacchia equilibrium point for  $(\mathbf{f}, \mathbf{K})$  whenever the sets  $K_i$  are compact; the proof is based on Begle's fixed point theorem for set-valued maps. More precisely, we have

**Theorem 4.2** *Let  $(M_i, g_i)$  be finite-dimensional Hadamard manifolds;  $K_i \subset M_i$  be non-empty, compact, geodesic convex sets; and  $U_i \subset M_i$  be open sets containing  $K_i$ ,  $i \in \{1, \dots, n\}$ . Assume that  $\mathbf{f} \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$  and  $\mathbf{K} \ni \mathbf{p} \mapsto \partial_C^\Delta \mathbf{f}(\mathbf{p})$  is upper semicontinuous. Then there exists at least one Nash-Stampacchia equilibrium point for  $(\mathbf{f}, \mathbf{K})$ , i.e.,  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$ .*

*Proof.* Fix  $\alpha > 0$  arbitrary. We prove that the set-valued map  $A_\alpha^{\mathbf{f}}$  has closed graph. Let  $(\mathbf{p}, \mathbf{q}) \in \mathbf{K} \times \mathbf{K}$  and the sequences  $\{\mathbf{p}_k\}, \{\mathbf{q}_k\} \subset \mathbf{K}$  such that  $\mathbf{q}_k \in A_\alpha^{\mathbf{f}}(\mathbf{p}_k)$  and  $(\mathbf{p}_k, \mathbf{q}_k) \rightarrow (\mathbf{p}, \mathbf{q})$  as  $k \rightarrow \infty$ . Then, for every  $k \in \mathbf{N}$ , there exists  $\xi_{C,k}^\Delta \in \partial_C^\Delta \mathbf{f}(\mathbf{p}_k)$  such that

$\mathbf{q}_k = P_{\mathbf{K}}(\exp_{\mathbf{p}_k}(-\alpha\xi_{C,k}^\Delta))$ . On account of Proposition 2.6 (i) $\Leftrightarrow$ (ii), the sequence  $\{\xi_{C,k}^\Delta\}$  is bounded on the cotangent bundle  $T^*\mathbf{M}$ . Using the identification between elements of the tangent and cotangent fibers, up to a subsequence, we may assume that  $\{\xi_{C,k}^\Delta\}$  converges to an element  $\xi_C^\Delta \in T_{\mathbf{p}}^*\mathbf{M}$ . Since the set-valued map  $\partial_C^\Delta \mathbf{f}$  is upper semicontinuous on  $\mathbf{K}$  and  $\mathbf{p}_k \rightarrow \mathbf{p}$  as  $k \rightarrow \infty$ , we have that  $\xi_C^\Delta \in \partial_C^\Delta \mathbf{f}(\mathbf{p})$ . The non-expansiveness of  $P_{\mathbf{K}}$  (see Proposition 2.1 (ii)) gives that

$$\begin{aligned} \mathbf{d}_{\mathbf{g}}(\mathbf{q}, P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^\Delta))) &\leq \mathbf{d}_{\mathbf{g}}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d}_{\mathbf{g}}(\mathbf{q}_k, P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^\Delta))) \\ &= \mathbf{d}_{\mathbf{g}}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d}_{\mathbf{g}}(P_{\mathbf{K}}(\exp_{\mathbf{p}_k}(-\alpha\xi_{C,k}^\Delta)), P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^\Delta))) \\ &\leq \mathbf{d}_{\mathbf{g}}(\mathbf{q}, \mathbf{q}_k) + \mathbf{d}_{\mathbf{g}}(\exp_{\mathbf{p}_k}(-\alpha\xi_{C,k}^\Delta), \exp_{\mathbf{p}}(-\alpha\xi_C^\Delta)) \end{aligned}$$

Letting  $k \rightarrow \infty$ , both terms in the last expression tend to zero. Indeed, the former follows from the fact that  $\mathbf{q}_k \rightarrow \mathbf{q}$  as  $k \rightarrow \infty$ , while the latter is a simple consequence of the local behaviour of the exponential map. Thus,

$$\mathbf{q} = P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\xi_C^\Delta)) \in P_{\mathbf{K}}(\exp_{\mathbf{p}}(-\alpha\partial_C^\Delta \mathbf{f}(\mathbf{p}))) = A_{\alpha}^{\mathbf{f}}(\mathbf{p}),$$

i.e., the graph of  $A_{\alpha}^{\mathbf{f}}$  is closed.

By definition, for each  $\mathbf{p} \in \mathbf{K}$  the set  $\partial_C^\Delta \mathbf{f}(\mathbf{p})$  is convex, so contractible. Since both  $P_{\mathbf{K}}$  and the exponential map are continuous,  $A_{\alpha}^{\mathbf{f}}(\mathbf{p})$  is contractible as well for each  $\mathbf{p} \in \mathbf{K}$ , so acyclic (see [27]).

Now, we are in position to apply Begle's fixed point theorem, see for instance McClendon [27, Proposition 1.1]. Consequently, there exists  $\mathbf{p} \in \mathbf{K}$  such that  $\mathbf{p} \in A_{\alpha}^{\mathbf{f}}(\mathbf{p})$ . On account of Theorem 4.1,  $\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ .  $\diamond$

**Remark 4.2** (a) Since  $\mathbf{f} \in \mathcal{L}(\mathbf{K}, \mathbf{U}, \mathbf{M})$  in Theorem 4.2, the partial Clarke gradients  $q \mapsto \partial_C f_i(\cdot, \mathbf{p}_{-i})(q)$  are upper semicontinuous,  $i \in \{1, \dots, n\}$ . However, in general, the diagonal Clarke subdifferential  $\partial_C^\Delta \mathbf{f}(\cdot)$  does not inherit this regularity property.

(b) Two unusual applications to Theorem 4.2 will be given in Examples 6.1 and 6.2; the first on the Poincaré disc, the second on the manifold of positive definite, symmetric matrices.

**4.3. Nash-Stampacchia equilibrium points; non-compact case.** In the sequel, we are focusing to the location of Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  in the case when  $K_i$  are *not* necessarily compact on the Hadamard manifolds  $(M_i, g_i)$ . Simple examples show that even the  $C^\infty$ -smoothness of the payoff functions are not enough to guarantee the existence of Nash(-Stampacchia) equilibria. Indeed, if  $f_1, f_2 : \mathbf{R}^2 \rightarrow \mathbf{R}$  are defined as  $f_1(x, y) = f_2(x, y) = e^{-x-y}$ , and  $K_1 = K_2 = [0, \infty)$ , then  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \emptyset$ . Therefore, in order to prove existence/location of Nash(-Stampacchia) equilibria on not necessarily compact strategy sets, one needs to require more specific assumptions on  $\mathbf{f} = (f_1, \dots, f_n)$ . Two such possible ways are described in the sequel.

The first existence result is based on a suitable coercivity assumption and Theorem 4.2. For a fixed  $\mathbf{p}_0 \in \mathbf{K}$ , we introduce the hypothesis:

$(H_{\mathbf{p}_0})$  There exists  $\xi_C^0 \in \partial_C^\Delta \mathbf{f}(\mathbf{p}_0)$  such that

$$L_{\mathbf{p}_0} = \limsup_{\mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{p}_0) \rightarrow \infty} \frac{\sup_{\xi_C \in \partial_C^\Delta \mathbf{f}(\mathbf{p})} \langle \xi_C, \exp_{\mathbf{p}}^{-1}(\mathbf{p}_0) \rangle_{\mathbf{g}} + \langle \xi_C^0, \exp_{\mathbf{p}_0}^{-1}(\mathbf{p}) \rangle_{\mathbf{g}}}{\mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{p}_0)} < -\|\xi_C^0\|_{\mathbf{g}}, \quad \mathbf{p} \in \mathbf{K}.$$

**Remark 4.3** (a) A similar assumption to hypothesis  $(H_{\mathbf{p}_0})$  can be found in Németh [31] in the context of variational inequalities.

(b) Note that for the above numerical example,  $(H_{\mathbf{p}_0})$  is not satisfied for any  $\mathbf{p}_0 = (x_0, y_0) \in [0, \infty) \times [0, \infty)$ . Indeed, one has  $L_{(x_0, y_0)} = -e^{x_0+y_0}$ , and  $\|\xi_C^0\|_{\mathbf{g}} = e^{x_0+y_0}\sqrt{2}$ . Therefore, the facts that  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \emptyset$  are not unexpected.

The precise statement of the existence result is as follows.

**Theorem 4.3** *Let  $(M_i, g_i)$  be finite-dimensional Hadamard manifolds;  $K_i \subset M_i$  be non-empty, closed, geodesic convex sets; and  $U_i \subset M_i$  be open sets containing  $K_i$ ,  $i \in \{1, \dots, n\}$ . Assume that  $\mathbf{f} \in \mathcal{L}(\mathbf{K}, \mathbf{U}, \mathbf{M})$ , the map  $\mathbf{K} \ni \mathbf{p} \mapsto \partial_C^\Delta \mathbf{f}(\mathbf{p})$  is upper semicontinuous, and hypothesis  $(H_{\mathbf{p}_0})$  holds for some  $\mathbf{p}_0 \in \mathbf{K}$ . Then there exists at least one Nash-Stampacchia equilibrium point for  $(\mathbf{f}, \mathbf{K})$ , i.e.,  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$ .*

*Proof.* Let  $E_0 \in \mathbf{R}$  such that  $L_{\mathbf{p}_0} < -E_0 < -\|\xi_C^0\|_{\mathbf{g}}$ . On account of hypothesis  $(H_{\mathbf{p}_0})$  there exists  $R > 0$  large enough such that for every  $\mathbf{p} \in \mathbf{K}$  with  $\mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{p}_0) \geq R$ , we have

$$\sup_{\xi_C \in \partial_C^\Delta \mathbf{f}(\mathbf{p})} \langle \xi_C, \exp_{\mathbf{p}}^{-1}(\mathbf{p}_0) \rangle_{\mathbf{g}} + \langle \xi_C^0, \exp_{\mathbf{p}_0}^{-1}(\mathbf{p}) \rangle_{\mathbf{g}} \leq -E_0 \mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{p}_0).$$

It is clear that  $\mathbf{K} \cap \overline{B}_{\mathbf{g}}(\mathbf{p}_0, R) \neq \emptyset$ , where  $\overline{B}_{\mathbf{g}}(\mathbf{p}_0, R)$  denotes the closed geodesic ball in  $(\mathbf{M}, \mathbf{g})$  with center  $\mathbf{p}_0$  and radius  $R$ . In particular, from (4) and (2), for every  $\mathbf{p} \in \mathbf{K}$  with  $\mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{p}_0) \geq R$ , the above relation yields

$$\begin{aligned} \sup_{\xi_C \in \partial_C^\Delta \mathbf{f}(\mathbf{p})} \langle \xi_C, \exp_{\mathbf{p}}^{-1}(\mathbf{p}_0) \rangle_{\mathbf{g}} &\leq -E_0 \mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{p}_0) + \|\xi_C^0\|_{\mathbf{g}} \|\exp_{\mathbf{p}_0}^{-1}(\mathbf{p})\|_{\mathbf{g}} \\ &= (-E_0 + \|\xi_C^0\|_{\mathbf{g}}) \mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{p}_0) \\ &< 0. \end{aligned} \tag{18}$$

Let  $\mathbf{K}_R = \mathbf{K} \cap \overline{B}_{\mathbf{g}}(\mathbf{p}_0, R)$ . It is clear that  $\mathbf{K}_R$  is a geodesic convex, compact subset of  $\mathbf{M}$ . By applying Theorem 4.2, we immediately have that  $\tilde{\mathbf{p}} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}_R) \neq \emptyset$ , i.e., there exists  $\tilde{\xi}_C \in \partial_C^\Delta \mathbf{f}(\tilde{\mathbf{p}})$  such that

$$\langle \tilde{\xi}_C, \exp_{\tilde{\mathbf{p}}}^{-1}(\mathbf{p}) \rangle_{\mathbf{g}} \geq 0 \quad \text{for all } \mathbf{p} \in \mathbf{K}_R. \tag{19}$$



It is also clear that  $\mathbf{d}_{\mathbf{g}}(\tilde{\mathbf{p}}, \mathbf{p}_0) < R$ . Indeed, assuming the contrary, we obtain from (18) that  $\langle \tilde{\xi}_C, \exp_{\tilde{\mathbf{p}}}^{-1}(\mathbf{p}_0) \rangle_{\mathbf{g}} < 0$ , which contradicts relation (19). Now, fix  $\mathbf{q} \in \mathbf{K}$  arbitrarily. Thus, for  $\varepsilon > 0$  small enough, the element  $\mathbf{p} = \exp_{\tilde{\mathbf{p}}}(\varepsilon \exp_{\tilde{\mathbf{p}}}^{-1}(\mathbf{q}))$  belongs both to  $\mathbf{K}$  and  $\overline{B}_{\mathbf{g}}(\mathbf{p}_0, R)$ , so  $\mathbf{K}_R$ . By substituting  $\mathbf{p}$  into (19), we obtain that  $\langle \tilde{\xi}_C, \exp_{\tilde{\mathbf{p}}}^{-1}(\mathbf{q}) \rangle_{\mathbf{g}} \geq 0$ . The arbitrariness of  $\mathbf{q} \in \mathbf{K}$  shows that  $\tilde{\mathbf{p}} \in \mathbf{K}$  is actually a Nash-Stampacchia equilibrium point for  $(\mathbf{f}, \mathbf{K})$ , which ends the proof.  $\diamond$

**Remark 4.4** A relevant application to Theorem 4.3 will be given in Example 6.3.

The second result in the non-compact case is based on a suitable Lipschitz-type assumption. In order to avoid technicalities in our further calculations, we will consider that  $\mathbf{f} \in \mathcal{C}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ . In this case,  $\partial_C^\Delta \mathbf{f}(\mathbf{p})$  and  $A_\alpha^\mathbf{f}(\mathbf{p})$  are singletons for every  $\mathbf{p} \in \mathbf{K}$  and  $\alpha > 0$ .

For  $\mathbf{f} \in \mathcal{C}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ ,  $\alpha > 0$  and  $0 < \rho < 1$  we introduce the hypothesis:

$$(H_{\mathbf{K}}^{\alpha, \rho}) \quad \mathbf{d}_{\mathbf{g}}(\exp_{\mathbf{p}}(-\alpha \partial_C^\Delta \mathbf{f}(\mathbf{p})), \exp_{\mathbf{q}}(-\alpha \partial_C^\Delta \mathbf{f}(\mathbf{q}))) \leq (1 - \rho) \mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{q}) \text{ for all } \mathbf{p}, \mathbf{q} \in \mathbf{K}.$$

**Remark 4.5** One can show that  $(H_{\mathbf{K}}^{\alpha, \rho})$  implies  $(H_{\mathbf{p}_0})$  for every  $\mathbf{p}_0 \in \mathbf{K}$  whenever  $(M_i, g_i)$  are Euclidean spaces. However, it is not clear if the same holds for Hadamard manifolds.

Finding fixed points for  $A_\alpha^\mathbf{f}$ , one could expect to apply dynamical systems; we consider both *discrete* and *continuous* ones. First, for some  $\alpha > 0$  and  $\mathbf{p}_0 \in \mathbf{M}$  fixed, we consider the discrete dynamical system

$$(DDS)_\alpha \quad \mathbf{p}_{k+1} = A_\alpha^\mathbf{f}(P_{\mathbf{K}}(\mathbf{p}_k)).$$

Second, according to Theorem 4.1, we clearly have that

$$\mathbf{p} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \Leftrightarrow 0 = \exp_{\mathbf{p}}^{-1}(A_\alpha^\mathbf{f}(\mathbf{p})) \text{ for all/some } \alpha > 0.$$

Consequently, for some  $\alpha > 0$  and  $\mathbf{p}_0 \in \mathbf{M}$  fixed, the above equivalence motivates the study of the continuous dynamical system

$$(CDS)_\alpha \quad \begin{cases} \dot{\eta}(t) = \exp_{\eta(t)}^{-1}(A_\alpha^\mathbf{f}(P_{\mathbf{K}}(\eta(t)))) \\ \eta(0) = \mathbf{p}_0. \end{cases}$$

The next result describes the exponential stability of the orbits in both cases.

**Theorem 4.4** *Let  $(M_i, g_i)$  be finite-dimensional Hadamard manifolds;  $K_i \subset M_i$  be non-empty, closed geodesics convex sets;  $U_i \subset M_i$  be open sets containing  $K_i$ ; and  $f_i : \mathbf{K} \rightarrow \mathbf{R}$  be functions,  $i \in \{1, \dots, n\}$  such that  $\mathbf{f} \in \mathcal{C}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ . Assume that  $(H_{\mathbf{K}}^{\alpha, \rho})$  holds true for some  $\alpha > 0$  and  $0 < \rho < 1$ . Then the set of Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  is a singleton, i.e.,  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{\tilde{\mathbf{p}}\}$ . Moreover, for each  $\mathbf{p}_0 \in \mathbf{M}$ , we have*

(i) the orbit  $\{\mathbf{p}_k\}$  of  $(DDS)_\alpha$  converges exponentially to  $\tilde{\mathbf{p}} \in \mathbf{K}$  and

$$\mathbf{d}_g(\mathbf{p}_k, \tilde{\mathbf{p}}) \leq \frac{(1-\rho)^k}{\rho} \mathbf{d}_g(\mathbf{p}_1, \mathbf{p}_0) \text{ for all } k \in \mathbf{N};$$

(ii) the orbit  $\eta$  of  $(CDS)_\alpha$  is globally defined on  $[0, \infty)$  and it converges exponentially to  $\tilde{\mathbf{p}} \in \mathbf{K}$  and

$$\mathbf{d}_g(\eta(t), \tilde{\mathbf{p}}) \leq e^{-\rho t} \mathbf{d}_g(\mathbf{p}_0, \tilde{\mathbf{p}}) \text{ for all } t \geq 0.$$

Furthermore, the set  $\mathbf{K}$  is invariant with respect to the orbits in both cases whenever  $\mathbf{p}_0 \in \mathbf{K}$ .

*Proof.* Let  $\mathbf{p}, \mathbf{q} \in \mathbf{M}$  be arbitrarily fixed. On account of the non-expansiveness of the projection  $P_{\mathbf{K}}$  (see Proposition 2.1 (ii)) and hypothesis  $(H_{\mathbf{K}}^{\alpha, \rho})$ , we have that

$$\begin{aligned} & \mathbf{d}_g((A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}})(\mathbf{p}), (A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}})(\mathbf{q})) \\ &= \mathbf{d}_g(P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\mathbf{p})}(-\alpha \partial_C^\Delta \mathbf{f}(P_{\mathbf{K}}(\mathbf{p})))), P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\mathbf{q})}(-\alpha \partial_C^\Delta \mathbf{f}(P_{\mathbf{K}}(\mathbf{q})))) \\ &\leq \mathbf{d}_g(\exp_{P_{\mathbf{K}}(\mathbf{p})}(-\alpha \partial_C^\Delta \mathbf{f}(P_{\mathbf{K}}(\mathbf{p}))), \exp_{P_{\mathbf{K}}(\mathbf{q})}(-\alpha \partial_C^\Delta \mathbf{f}(P_{\mathbf{K}}(\mathbf{q})))) \\ &\leq (1-\rho) \mathbf{d}_g(P_{\mathbf{K}}(\mathbf{p}), P_{\mathbf{K}}(\mathbf{q})) \\ &\leq (1-\rho) \mathbf{d}_g(\mathbf{p}, \mathbf{q}), \end{aligned}$$

which means that the map  $A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}} : \mathbf{M} \rightarrow \mathbf{M}$  is a  $(1-\rho)$ -contraction on  $\mathbf{M}$ .

(i) Since  $(\mathbf{M}, \mathbf{d}_g)$  is a complete metric space, a standard Banach fixed point argument shows that  $A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}}$  has a unique fixed point  $\tilde{\mathbf{p}} \in \mathbf{M}$ . Since  $\text{Im} A_\alpha^{\mathbf{f}} \subset \mathbf{K}$ , then  $\tilde{\mathbf{p}} \in \mathbf{K}$ . Therefore, we have that  $A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}}) = \tilde{\mathbf{p}}$ . Due to Theorem 4.1,  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{\tilde{\mathbf{p}}\}$  and the estimate for  $\mathbf{d}_g(\mathbf{p}_k, \tilde{\mathbf{p}})$  yields in a usual manner.

(ii) Since  $A_\alpha^{\mathbf{f}} \circ P_{\mathbf{K}} : \mathbf{M} \rightarrow \mathbf{M}$  is a  $(1-\rho)$ -contraction on  $\mathbf{M}$  (thus locally Lipschitz in particular), the map  $\mathbf{M} \ni \mathbf{p} \mapsto G(\mathbf{p}) := \exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})))$  is of class  $C^{1-0}$ . Now, we may guarantee the existence of a unique maximal orbit  $\eta : [0, T_{\max}) \rightarrow \mathbf{M}$  of  $(CDS)_\alpha$ .

We assume that  $T_{\max} < \infty$ . Let us consider the Lyapunov function  $h : [0, T_{\max}) \rightarrow \mathbf{R}$  defined by

$$h(t) = \frac{1}{2} \mathbf{d}_g^2(\eta(t), \tilde{\mathbf{p}}).$$

The function  $h$  is differentiable for a.e.  $t \in [0, T_{\max})$  and in the differentiable points of  $\eta$  we have

$$\begin{aligned} h'(t) &= -\mathbf{g}(\dot{\eta}(t), \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \\ &= -\mathbf{g}(\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))), \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \quad (\text{cf. } (CDS)_\alpha) \\ &= -\mathbf{g}(\exp_{\eta(t)}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}), \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \end{aligned}$$

$$\begin{aligned}
& -\mathbf{g}(\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}), \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})) \\
& \leq \|\exp_{\eta(t)}^{-1}(A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}} \cdot \|\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}} - \|\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}}^2.
\end{aligned}$$

In the last estimate we used the Cauchy-Schwartz inequality (2). From (4) we have that

$$\|\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}} = \mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}). \quad (20)$$

We claim that for every  $t \in [0, T_{\max})$  one has

$$\|\exp_{\eta(t)}^{-1}(A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}} \leq \mathbf{d}_{\mathbf{g}}(A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}). \quad (21)$$

To see this, fix a point  $t \in [0, T_{\max})$  where  $\eta$  is differentiable, and let  $\gamma : [0, 1] \rightarrow \mathbf{M}$ ,  $\tilde{\gamma} : [0, 1] \rightarrow T_{\eta(t)}\mathbf{M}$  and  $\overline{\gamma} : [0, 1] \rightarrow T_{\eta(t)}\mathbf{M}$  be three curves such that

- $\gamma$  is the unique minimal geodesic joining the two points  $\gamma(0) = \tilde{\mathbf{p}} \in \mathbf{K}$  and  $\gamma(1) = A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))$ ;
- $\tilde{\gamma}(s) = \exp_{\eta(t)}^{-1}(\gamma(s))$ ,  $s \in [0, 1]$ ;
- $\overline{\gamma}(s) = (1-s)\exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}}) + s\exp_{\eta(t)}^{-1}(A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))))$ ,  $s \in [0, 1]$ .

By the definition of  $\gamma$ , we have that

$$L_{\mathbf{g}}(\gamma) = \mathbf{d}_{\mathbf{g}}(A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}). \quad (22)$$

Moreover, since  $\overline{\gamma}$  is a segment of the straight line in  $T_{\eta(t)}\mathbf{M}$  that joins the endpoints of  $\tilde{\gamma}$ , we have that

$$l(\overline{\gamma}) \leq l(\tilde{\gamma}). \quad (23)$$

Here,  $l$  denotes the length function on  $T_{\eta(t)}\mathbf{M}$ . Moreover, since the curvature of  $(\mathbf{M}, \mathbf{g})$  is non-positive, we may apply a Rauch-type comparison result for the lengths of  $\gamma$  and  $\tilde{\gamma}$ , see do Carmo [14, Proposition 2.5, p.218], obtaining that

$$l(\tilde{\gamma}) \leq L_{\mathbf{g}}(\gamma). \quad (24)$$

Combining relations (22), (23) and (24) with the fact that

$$l(\overline{\gamma}) = \|\exp_{\eta(t)}^{-1}(A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t)))) - \exp_{\eta(t)}^{-1}(\tilde{\mathbf{p}})\|_{\mathbf{g}},$$

relation (21) holds true.

Coming back to  $h'(t)$ , in view of (20) and (21), it turns out that

$$h'(t) \leq \mathbf{d}_{\mathbf{g}}(A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}) \cdot \mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}) - \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}). \quad (25)$$

On the other hand, note that  $\tilde{\mathbf{p}} \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ , i.e.,  $A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}}) = \tilde{\mathbf{p}}$ . By exploiting the non-expansiveness of the projection operator  $P_{\mathbf{K}}$ , see Proposition 2.1 (ii), and  $(H_{\mathbf{K}}^{\alpha, \rho})$ , we have that

$$\begin{aligned}
\mathbf{d}_{\mathbf{g}}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), \tilde{\mathbf{p}}) &= \mathbf{d}_{\mathbf{g}}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\eta(t))), A_\alpha^{\mathbf{f}}(\tilde{\mathbf{p}})) \\
&= \mathbf{d}_{\mathbf{g}}(P_{\mathbf{K}}(\exp_{P_{\mathbf{K}}(\eta(t))}(-\alpha \partial_C^\Delta \mathbf{f}(P_{\mathbf{K}}(\eta(t))))), P_{\mathbf{K}}(\exp_{\tilde{\mathbf{p}}}(-\alpha \partial_C^\Delta \mathbf{f}(\tilde{\mathbf{p}})))) \\
&\leq \mathbf{d}_{\mathbf{g}}(\exp_{P_{\mathbf{K}}(\eta(t))}(-\alpha \partial_C^\Delta \mathbf{f}(P_{\mathbf{K}}(\eta(t)))), \exp_{\tilde{\mathbf{p}}}(-\alpha \partial_C^\Delta \mathbf{f}(\tilde{\mathbf{p}}))) \\
&\leq (1 - \rho) \mathbf{d}_{\mathbf{g}}(P_{\mathbf{K}}(\eta(t)), \tilde{\mathbf{p}}) \\
&= (1 - \rho) \mathbf{d}_{\mathbf{g}}(P_{\mathbf{K}}(\eta(t)), P_{\mathbf{K}}(\tilde{\mathbf{p}})) \\
&\leq (1 - \rho) \mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}).
\end{aligned}$$

Combining the above relation with (25), for a.e.  $t \in [0, T_{\max})$  it yields

$$h'(t) \leq (1 - \rho) \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}) - \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}) = -\rho \mathbf{d}_{\mathbf{g}}^2(\eta(t), \tilde{\mathbf{p}}),$$

which is nothing but

$$h'(t) \leq -2\rho h(t) \quad \text{for a.e. } t \in [0, T_{\max}).$$

Due to the latter inequality, we have that

$$\frac{d}{dt}[h(t)e^{2\rho t}] = [h'(t) + 2\rho h(t)]e^{2\rho t} \leq 0 \quad \text{for a.e. } t \in [0, T_{\max}).$$

After integration, one gets

$$h(t)e^{2\rho t} \leq h(0) \quad \text{for all } t \in [0, T_{\max}). \quad (26)$$

According to (26), the function  $h$  is bounded on  $[0, T_{\max})$ ; thus, there exists  $\bar{\mathbf{p}} \in \mathbf{M}$  such that  $\lim_{t \nearrow T_{\max}} \eta(t) = \bar{\mathbf{p}}$ . The last limit means that  $\eta$  can be extended toward the value  $T_{\max}$ , which contradicts the maximality of  $T_{\max}$ . Thus,  $T_{\max} = \infty$ .

Now, relation (26) leads to the required estimate; indeed, we have

$$\mathbf{d}_{\mathbf{g}}(\eta(t), \tilde{\mathbf{p}}) \leq e^{-\rho t} \mathbf{d}_{\mathbf{g}}(\eta(0), \tilde{\mathbf{p}}) = e^{-\rho t} \mathbf{d}_{\mathbf{g}}(\mathbf{p}_0, \tilde{\mathbf{p}}) \quad \text{for all } t \in [0, \infty),$$

which concludes the proof of (ii).

Now, we assume that  $\mathbf{p}_0 \in \mathbf{K}$  and we are dealing with the viability of the orbits for problems  $(DDS)_\alpha$  and  $(CDS)_\alpha$ , respectively. First, since  $\text{Im} A_\alpha^{\mathbf{f}} \subset \mathbf{K}$ , then the orbit of  $(DDS)_\alpha$  belongs to  $\mathbf{K}$ , i.e.,  $\mathbf{p}_k \in \mathbf{K}$  for every  $k \in \mathbf{N}$ . Second, we shall prove that  $\mathbf{K}$  is invariant with respect to the solutions of  $(CDS)_\alpha$ , i.e., the image of the global solution  $\eta : [0, \infty) \rightarrow \mathbf{M}$  of  $(CDS)_\alpha$  with  $\eta(0) = \mathbf{p}_0 \in \mathbf{K}$ , entirely belongs to the set  $\mathbf{K}$ . To show the latter fact, we are going to apply Proposition 2.8 by choosing  $M := \mathbf{M}$  and  $G : \mathbf{M} \rightarrow T\mathbf{M}$  defined by  $G(\mathbf{p}) := \exp_{\mathbf{p}}^{-1}(A_\alpha^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})))$ .

Fix  $\mathbf{p} \in \mathbf{K}$  and  $\xi \in N_F(\mathbf{p}; \mathbf{K})$ . Since  $\mathbf{K}$  is geodesic convex in  $(\mathbf{M}, \mathbf{g})$ , on account of Theorem 1, we have that  $\langle \xi, \exp_{\mathbf{p}}^{-1}(\mathbf{q}) \rangle_{\mathbf{g}} \leq 0$  for all  $\mathbf{q} \in \mathbf{K}$ . In particular, if we choose  $\mathbf{q}_0 = A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p})) \in \mathbf{K}$ , it turns out that

$$H_G(\mathbf{p}, \xi) = \langle \xi, G(\mathbf{p}) \rangle_{\mathbf{g}} = \langle \xi, \exp_{\mathbf{p}}^{-1}(A_{\alpha}^{\mathbf{f}}(P_{\mathbf{K}}(\mathbf{p}))) \rangle_{\mathbf{g}} = \langle \xi, \exp_{\mathbf{p}}^{-1}(\mathbf{q}_0) \rangle_{\mathbf{g}} \leq 0.$$

Consequently, our claim is proved by applying Proposition 2.8.  $\diamond$

## 5 Curvature rigidity: metric projections versus Hadamard manifolds

The obtuse-angle property and the non-expansiveness of  $P_{\mathbf{K}}$  for the closed, geodesic convex set  $\mathbf{K} \subset \mathbf{M}$  played indispensable roles in the proof of Theorems 4.1-4.4, which are well-known features of Hadamard manifolds (see Proposition 2.1). In Section 4 the product manifold  $(\mathbf{M}, \mathbf{g})$  is considered to be a Hadamard one due to the fact that  $(M_i, g_i)$  are Hadamard manifolds themselves for each  $i \in \{1, \dots, n\}$ . We actually have the following characterization which is also of geometric interests in its own right and entitles us to assert that Hadamard manifolds are the natural framework to develop the theory of Nash-Stampacchia equilibria on manifolds.

**Theorem 5.1** *Let  $(M_i, g_i)$  be complete, simply connected Riemannian manifolds,  $i \in \{1, \dots, n\}$ , and  $(\mathbf{M}, \mathbf{g})$  their product manifold. The following statements are equivalent:*

- (i) *Any non-empty, closed, geodesic convex set  $\mathbf{K} \subset \mathbf{M}$  verifies the obtuse-angle property and  $P_{\mathbf{K}}$  is non-expansive;*
- (ii)  *$(M_i, g_i)$  are Hadamard manifolds for every  $i \in \{1, \dots, n\}$ .*

*Proof.* (ii) $\Rightarrow$ (i). As mentioned before, if  $(M_i, g_i)$  are Hadamard manifolds for every  $i \in \{1, \dots, n\}$ , then  $(\mathbf{M}, \mathbf{g})$  is also a Hadamard manifold, see Ballmann [2, Example 4, p.147] and O'Neill [32, Lemma 40, p. 209]. It remains to apply Proposition 2.1 for the Hadamard manifold  $(\mathbf{M}, \mathbf{g})$ .

(i) $\Rightarrow$ (ii). We first prove that  $(\mathbf{M}, \mathbf{g})$  is a Hadamard manifold. Since  $(M_i, g_i)$  are complete and simply connected Riemannian manifolds for every  $i \in \{1, \dots, n\}$ , the same is true for  $(\mathbf{M}, \mathbf{g})$ . We now show that the sectional curvature of  $(\mathbf{M}, \mathbf{g})$  is non-positive. To see this, let  $\mathbf{p} \in \mathbf{M}$  and  $\mathbf{W}_0, \mathbf{V}_0 \in T_{\mathbf{p}}\mathbf{M} \setminus \{\mathbf{0}\}$ . We claim that the sectional curvature of the two-dimensional subspace  $S = \text{span}\{\mathbf{W}_0, \mathbf{V}_0\} \subset T_{\mathbf{p}}\mathbf{M}$  at the point  $\mathbf{p}$  is non-positive, i.e.,  $K_{\mathbf{p}}(S) \leq 0$ . We assume without loosing the generality that  $\mathbf{V}_0$  and  $\mathbf{W}_0$  are  $\mathbf{g}$ -perpendicular, i.e.,  $\mathbf{g}(\mathbf{W}_0, \mathbf{V}_0) = 0$ .

Let us fix  $r_{\mathbf{p}} > 0$  and  $\delta > 0$  such that  $B_{\mathbf{g}}(\mathbf{p}, r_{\mathbf{p}})$  is a totally normal ball of  $\mathbf{p}$  and

$$\delta (\|\mathbf{W}_0\|_{\mathbf{g}} + 2\|\mathbf{V}_0\|_{\mathbf{g}}) < r_{\mathbf{p}}. \tag{27}$$

Let  $\sigma : [-\delta, 2\delta] \rightarrow \mathbf{M}$  be the geodesic segment  $\sigma(t) = \exp_{\mathbf{p}}(t\mathbf{V}_0)$  and  $\mathbf{W}$  be the unique parallel vector field along  $\sigma$  with the initial data  $\mathbf{W}(0) = \mathbf{W}_0$ . For any  $t \in [0, \delta]$ , let  $\gamma_t : [0, \delta] \rightarrow \mathbf{M}$  be the geodesic segment  $\gamma_t(u) = \exp_{\sigma(t)}(u\mathbf{W}(t))$ .

Let us fix  $t, u \in [0, \delta]$  arbitrarily,  $u \neq 0$ . Due to (27), the geodesic segment  $\gamma_t|_{[0, u]}$  belongs to the totally normal ball  $B_{\mathbf{g}}(\mathbf{p}, r_{\mathbf{p}})$  of  $\mathbf{p}$ ; thus,  $\gamma_t|_{[0, u]}$  is the unique minimal geodesic joining the point  $\gamma_t(0) = \sigma(t)$  to  $\gamma_t(u)$ . Moreover, since  $\mathbf{W}$  is the parallel transport of  $\mathbf{W}(0) = \mathbf{W}_0$  along  $\sigma$ , we have  $\mathbf{g}(\mathbf{W}(t), \dot{\sigma}(t)) = \mathbf{g}(\mathbf{W}(0), \dot{\sigma}(0)) = \mathbf{g}(\mathbf{W}_0, \mathbf{V}_0) = 0$ ; therefore,

$$\mathbf{g}(\dot{\gamma}_t(0), \dot{\sigma}(t)) = \mathbf{g}(\mathbf{W}(t), \dot{\sigma}(t)) = 0.$$

Consequently, the minimal geodesic segment  $\gamma_t|_{[0, u]}$  joining  $\gamma_t(0) = \sigma(t)$  to  $\gamma_t(u)$ , and the set  $\mathbf{K} = \text{Im}\sigma = \{\sigma(t) : t \in [-\delta, 2\delta]\}$  fulfill hypothesis  $(OA_2)$ . Note that  $\text{Im}\sigma$  is a closed, geodesic convex set in  $\mathbf{M}$ ; thus, from hypothesis (i) it follows that  $\text{Im}\sigma$  verifies the obtuse-angle property and  $P_{\text{Im}\sigma}$  is non-expansive. Thus,  $(OA_2)$  implies  $(OA_1)$ , i.e., for every  $t, u \in [0, \delta]$ , we have  $\sigma(t) \in P_{\text{Im}\sigma}(\gamma_t(u))$ . Since  $\text{Im}\sigma$  is a Chebyshev set (cf. the non-expansiveness of  $P_{\text{Im}\sigma}$ ), for every  $t, u \in [0, \delta]$ , we have

$$P_{\text{Im}\sigma}(\gamma_t(u)) = \{\sigma(t)\}. \quad (28)$$

Thus, for every  $t, u \in [0, \delta]$ , relation (28) and the non-expansiveness of  $P_{\text{Im}\sigma}$  imply

$$\begin{aligned} \mathbf{d}_{\mathbf{g}}(\mathbf{p}, \sigma(t)) &= \mathbf{d}_{\mathbf{g}}(\sigma(0), \sigma(t)) = \mathbf{d}_{\mathbf{g}}(P_{\text{Im}\sigma}(\gamma_0(u)), P_{\text{Im}\sigma}(\gamma_t(u))) \\ &\leq \mathbf{d}_{\mathbf{g}}(\gamma_0(u), \gamma_t(u)). \end{aligned} \quad (29)$$

The above construction (i.e., the parallel transport of  $\mathbf{W}(0) = \mathbf{W}_0$  along  $\sigma$ ) and the formula of the sectional curvature in the parallelogramoid of Levi-Civita defined by the points  $\mathbf{p}$ ,  $\sigma(t)$ ,  $\gamma_0(u)$ ,  $\gamma_t(u)$  give

$$K_{\mathbf{p}}(S) = \lim_{u, t \rightarrow 0} \frac{\mathbf{d}_{\mathbf{g}}^2(\mathbf{p}, \sigma(t)) - \mathbf{d}_{\mathbf{g}}^2(\gamma_0(u), \gamma_t(u))}{\mathbf{d}_{\mathbf{g}}(\mathbf{p}, \gamma_0(u)) \cdot \mathbf{d}_{\mathbf{g}}(\mathbf{p}, \sigma(t))}.$$

According to (29), the latter limit is non-positive, so  $K_{\mathbf{p}}(S) \leq 0$ , which concludes the first part, namely,  $(\mathbf{M}, \mathbf{g})$  is a Hadamard manifold.

Now, the main result of Chen [10, Theorem 1] implies that the metric spaces  $(M_i, d_{g_i})$  are Aleksandrov NPC spaces for every  $i \in \{1, \dots, n\}$ . Consequently, for each  $i \in \{1, \dots, n\}$ , the Riemannian manifolds  $(M_i, g_i)$  have non-positive sectional curvature, thus they are Hadamard manifolds. The proof is complete.  $\diamond$

**Remark 5.1** The obtuse-angle property and the non-expansiveness of the metric projection are also key tools behind the theory of monotone vector fields, proximal point algorithms and variational inequalities developed on Hadamard manifolds; see Li, López and Martín-Márquez [24], [25], and Németh [31]. Within the class of Riemannian manifolds, Theorem 5.1 shows in particular that Hadamard manifolds are indeed the appropriate

frameworks for developing successfully the approaches in [24], [25], and [31] and further related works.

## 6 Examples

In this section we present various examples where our main results can be efficiently applied; for convenience, we give all the details in our calculations by keeping also the notations from the previous sections.

**Example 6.1** Let

$$K_1 = \{(x_1, x_2) \in \mathbf{R}_+^2 : x_1^2 + x_2^2 \leq 4 \leq (x_1 - 1)^2 + x_2^2\}, \quad K_2 = [-1, 1],$$

and the functions  $f_1, f_2 : K_1 \times K_2 \rightarrow \mathbf{R}$  defined for  $(x_1, x_2) \in K_1$  and  $y \in K_2$  by

$$f_1((x_1, x_2), y) = y(x_1^3 + y(1 - x_2)^3); \quad f_2((x_1, x_2), y) = -y^2 x_2 + 4|y|(x_1 + 1).$$

It is clear that  $K_1 \subset \mathbf{R}^2$  is not convex in the usual sense while  $K_2 \subset \mathbf{R}$  is. However, if we consider the Poincaré upper-plane model  $(\mathbf{H}^2, g_{\mathbf{H}})$ , the set  $K_1 \subset \mathbf{H}^2$  is geodesic convex (and compact) with respect to the metric  $g_{\mathbf{H}} = (\frac{\delta_{ij}}{x_2^2})$ . Therefore, we embed the set  $K_1$  into the Hadamard manifold  $(\mathbf{H}^2, g_{\mathbf{H}})$ , and  $K_2$  into the standard Euclidean space  $(\mathbf{R}, g_0)$ . After natural extensions of  $f_1(\cdot, y)$  and  $f_2((x_1, x_2), \cdot)$  to the whole  $U_1 = \mathbf{H}^2$  and  $U_2 = \mathbf{R}$ , respectively, we clearly have that  $f_1(\cdot, y)$  is a  $C^1$  function on  $\mathbf{H}^2$  for every  $y \in K_2$ , while  $f_2((x_1, x_2), \cdot)$  is a locally Lipschitz function on  $\mathbf{R}$  for every  $(x_1, x_2) \in K_1$ . Thus,  $\mathbf{f} = (f_1, f_2) \in \mathcal{L}_{(K_1 \times K_2, \mathbf{H}^2 \times \mathbf{R}, \mathbf{H}^2 \times \mathbf{R})}$  and for every  $((x_1, x_2), y) \in \mathbf{K} = K_1 \times K_2$ , we have

$$\begin{aligned} \partial_C^1 f_1((x_1, x_2), y) &= \text{grad} f_1(\cdot, y)(x_1, x_2) = \left( g_{\mathbf{H}}^{ij} \frac{\partial f_1(\cdot, y)}{\partial x_j} \right)_i = 3yx_2^2(x_1^2, -y(1 - x_2)^2); \\ \partial_C^2 f_2((x_1, x_2), y) &= \begin{cases} -2yx_2 - 4(x_1 + 1) & \text{if } y < 0, \\ 4(x_1 + 1)[-1, 1] & \text{if } y = 0, \\ -2yx_2 + 4(x_1 + 1) & \text{if } y > 0. \end{cases} \end{aligned}$$

It is now clear that the map  $\mathbf{K} \ni ((x_1, x_2), y) \mapsto \partial_C^{\Delta} \mathbf{f}(((x_1, x_2), y))$  is upper semicontinuous. Consequently, on account of Theorem 4.2,  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$ , and its elements are precisely the solutions  $((\tilde{x}_1, \tilde{x}_2), \tilde{y}) \in \mathbf{K}$  of the system

$$\begin{cases} \langle \partial_C^1 f_1((\tilde{x}_1, \tilde{x}_2), \tilde{y}), \exp_{(\tilde{x}_1, \tilde{x}_2)}^{-1}(q_1, q_2) \rangle_{g_{\mathbf{H}}} \geq 0 & \text{for all } (q_1, q_2) \in K_1, \\ \xi_C^2(q - \tilde{y}) \geq 0 \text{ for some } \xi_C^2 \in \partial_C^2 f_2((\tilde{x}_1, \tilde{x}_2), \tilde{y}) & \text{for all } q \in K_2. \end{cases} \quad (S_1)$$

In order to solve  $(S_1)$  we first observe that

$$K_1 \subset \{(x_1, x_2) \in \mathbf{R}^2 : \sqrt{3} \leq x_2 \leq 2(x_1 + 1)\}. \quad (30)$$

We distinguish four cases:

(a) If  $\tilde{y} = 0$  then both inequalities of  $(S_1)$  hold for every  $(\tilde{x}_1, \tilde{x}_2) \in K_1$  by choosing  $\xi_C^2 = 0 \in \partial_C^2 f_2((\tilde{x}_1, \tilde{x}_2), 0)$  in the second relation. Thus,  $((\tilde{x}_1, \tilde{x}_2), 0) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$  for every  $(\tilde{x}_1, \tilde{x}_2) \in \mathbf{K}$ .

(b) Let  $0 < \tilde{y} < 1$ . The second inequality of  $(S_1)$  gives that  $-2\tilde{y}\tilde{x}_2 + 4(\tilde{x}_1 + 1) = 0$ ; together with (30) it yields  $0 = \tilde{y}\tilde{x}_2 - 2(\tilde{x}_1 + 1) < \tilde{x}_2 - 2(\tilde{x}_1 + 1) \leq 0$ , a contradiction.

(c) Let  $\tilde{y} = 1$ . The second inequality of  $(S_1)$  is true if and only if  $-2\tilde{x}_2 + 4(\tilde{x}_1 + 1) \leq 0$ . Due to (30), we necessarily have  $\tilde{x}_2 = 2(\tilde{x}_1 + 1)$ ; this Euclidean line intersects the set  $K_1$  in the unique point  $(\tilde{x}_1, \tilde{x}_2) = (0, 2) \in K_1$ . By the geometrical meaning of the exponential map one can conclude that

$$\{t \exp_{(0,2)}^{-1}(q_1, q_2) : (q_1, q_2) \in K_1, t \geq 0\} = \{(x, -y) \in \mathbf{R}^2 : x, y \geq 0\}.$$

Taking into account this relation and  $\partial_C^1 f_1((0, 2), 1) = (0, -12)$ , the first inequality of  $(S_1)$  holds true as well. Therefore,  $((0, 2), 1) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ .

(d) Similar reason as in (b) (for  $-1 < \tilde{y} < 0$ ) and (c) (for  $\tilde{y} = -1$ ) gives that  $((0, 2), -1) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ . Thus, from (a)-(d) we have that

$$\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = (K_1 \times \{0\}) \cup \{((0, 2), 1), ((0, 2), -1)\}.$$

Now, on account of Theorem 3.1 (i) we may easily select the Nash equilibrium points for  $(\mathbf{f}, \mathbf{K})$  among the elements of  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$  obtaining that

$$\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = K_1 \times \{0\}.$$

◇

In the rest of the paper we deal with some applications involving matrices; thus, we recall some basic notions from the matrix-calculus. Fix  $n \geq 2$ . Let  $M_n(\mathbf{R})$  be the set of symmetric  $n \times n$  matrices with real values, and  $M_n^+(\mathbf{R}) \subset M_n(\mathbf{R})$  be the cone of symmetric positive definite matrices. The standard inner product on  $M_n(\mathbf{R})$  is defined as

$$\langle U, V \rangle = \text{tr}(UV). \tag{31}$$

Here,  $\text{tr}(Y)$  denotes the trace of  $Y \in M_n(\mathbf{R})$ . It is well-known that  $(M_n(\mathbf{R}), \langle \cdot, \cdot \rangle)$  is an Euclidean space, the unique geodesic between  $X, Y \in M_n(\mathbf{R})$  is

$$\gamma_{X,Y}^E(s) = (1-s)X + sY, \quad s \in [0, 1]. \tag{32}$$

The set  $M_n^+(\mathbf{R})$  will be endowed with the Killing form

$$\langle\langle U, V \rangle\rangle_X = \text{tr}(X^{-1}VX^{-1}U), \quad X \in M_n^+(\mathbf{R}), \quad U, V \in T_X(M_n^+(\mathbf{R})). \tag{33}$$



Note that the pair  $(M_n^+(\mathbf{R}), \langle\langle \cdot, \cdot \rangle\rangle)$  is a Hadamard manifold, see Lang [22, Chapter XII], and  $T_X(M_n^+(\mathbf{R})) \simeq M_n(\mathbf{R})$ . The unique geodesic segment connecting  $X, Y \in M_n^+(\mathbf{R})$  is defined by

$$\gamma_{X,Y}^H(s) = X^{1/2}(X^{-1/2}YX^{-1/2})^sX^{1/2}, \quad s \in [0, 1]. \quad (34)$$

In particular,  $\frac{d}{ds}\gamma_{X,Y}^H(s)|_{s=0} = X^{1/2} \ln(X^{-1/2}YX^{-1/2})X^{1/2}$ ; consequently, for each  $X, Y \in M_n^+(\mathbf{R})$ , we have

$$\exp_X^{-1} Y = X^{1/2} \ln(X^{-1/2}YX^{-1/2})X^{1/2}.$$

Moreover, the metric function on  $M_n^+(\mathbf{R})$  is given by

$$d_H^2(X, Y) = \langle\langle \exp_X^{-1} Y, \exp_X^{-1} Y \rangle\rangle_X = \text{tr}(\ln^2(X^{-1/2}YX^{-1/2})). \quad (35)$$

**Example 6.2** Let

$$K_1 = [0, 2], \quad K_2 = \{X \in M_n^+(\mathbf{R}) : \text{tr}(\ln^2 X) \leq 1 \leq \det X \leq 2\},$$

and the functions  $f_1, f_2 : K_1 \times K_2 \rightarrow \mathbf{R}$  defined by

$$f_1(t, X) = (\max(t, 1))^{n-1} \text{tr}^2(X) - 4n \ln(t+1) S_2(X), \quad (36)$$

$$f_2(t, X) = g(t) \left( \text{tr}(X^{-1}) + 1 \right)^{t+1} + h(t) \ln \det X. \quad (37)$$

Here,  $S_2(Y)$  denotes the second elementary symmetric function of the eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $Y$ , i.e.,

$$S_2(Y) = \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \lambda_{i_2}, \quad (38)$$

and  $g, h : K_1 \rightarrow \mathbf{R}$  are two continuous functions such that

$$h(t) \geq 2(n+1)g(t) \geq 0 \text{ for all } t \in K_1. \quad (39)$$

The elements of  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$  are the solutions  $(\tilde{t}, \tilde{X}) \in \mathbf{K}$  of the system

$$\begin{cases} [(\max(t, 1))^{n-1} - (\max(\tilde{t}, 1))^{n-1}] \text{tr}^2(\tilde{X}) \geq 4n S_2(\tilde{X}) \ln \frac{t+1}{\tilde{t}+1}, & \forall t \in K_1, \\ g(\tilde{t}) \left[ (\text{tr}(Y^{-1}) + 1)^{\tilde{t}+1} - (\text{tr}(\tilde{X}^{-1}) + 1)^{\tilde{t}+1} \right] + h(\tilde{t}) \ln \frac{\det Y}{\det \tilde{X}} \geq 0, & \forall Y \in K_2. \end{cases} \quad (S_2)$$

The involved forms in  $(S_2)$  suggest an approach via the Nash-Stampacchia equilibria for  $(\mathbf{f}, \mathbf{K})$ ; first of all, we have to find the appropriate context where the machinery described in §4 works efficiently.

At first glance, the natural geometric framework seems to be  $M_n(\mathbf{R})$  with the inner product  $\langle \cdot, \cdot \rangle$  defined in (31). Note however that the set  $K_2$  is not geodesic convex with

respect to  $\langle \cdot, \cdot \rangle$ . Indeed, let  $X = \text{diag}(2, 1, \dots, 1) \in K_2$  and  $Y = \text{diag}(1, 2, \dots, 1) \in K_2$  and  $\gamma_{X,Y}^E$  be the Euclidean geodesic connecting them, see (32); although  $\gamma_{X,Y}^E(s) \in M_n^+(\mathbf{R})$  and  $\text{tr}(\ln^2(\gamma_{X,Y}^E(s))) = \ln^2(2-s) + \ln^2(1+s) \leq \ln^2 2$  for every  $s \in [0, 1]$ , we have that  $\det(\gamma_{X,Y}^E(s)) > 2$  for every  $0 < s < 1$ . Consequently, a more appropriate metric is needed to provide some sort of geodesic convexity for  $K_2$ . To complete this fact, we restrict our attention to the cone of symmetric positive definite matrices  $M_n^+(\mathbf{R})$  with the metric introduced in (33).

Let  $I_n \in M_n^+(\mathbf{R})$  be the identity matrix, and  $\overline{B}_H(I_n, 1)$  be the closed geodesic ball in  $M_n^+(\mathbf{R})$  with center  $I_n$  and radius 1. Note that

$$K_2 = \overline{B}_H(I_n, 1) \cap \{X \in M_n^+(\mathbf{R}) : 1 \leq \det X \leq 2\}.$$

Indeed, for every  $X \in M_n^+(\mathbf{R})$ , we have

$$d_H^2(I_n, X) = \text{tr}(\ln^2 X). \quad (40)$$

Since  $K_2$  is bounded and closed, on account of the Hopf-Rinow theorem,  $K_2$  is compact. Moreover, as a geodesic ball in the Hadamard manifold  $(M_n^+(\mathbf{R}), \langle \langle \cdot, \cdot \rangle \rangle)$ , the set  $\overline{B}_H(I_n, 1)$  is geodesic convex. Keeping the notation from (34), if  $X, Y \in K_2$ , one has for every  $s \in [0, 1]$  that

$$\det(\gamma_{X,Y}^H(s)) = (\det X)^{1-s}(\det Y)^s \in [1, 2],$$

which shows the geodesic convexity of  $K_2$  in  $(M_n^+(\mathbf{R}), \langle \langle \cdot, \cdot \rangle \rangle)$ .

After naturally extending the functions  $f_1(\cdot, X)$  and  $f_2(t, \cdot)$  to  $U_1 = (-\frac{1}{2}, \infty)$  and  $U_2 = M_n^+(\mathbf{R})$  by the same expressions (see (36) and (37)), we clearly have that  $\mathbf{f} = (f_1, f_2) \in \mathcal{L}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ , where  $\mathbf{U} = U_1 \times U_2$ , and  $\mathbf{M} = \mathbf{R} \times M_n^+(\mathbf{R})$ . A standard computation shows that for every  $(t, X) \in U_1 \times K_2$ , we have

$$\partial_C^1 f_1(t, X) = -\frac{4nS_2(X)}{t+1} + \text{tr}^2(X) \cdot \begin{cases} 0 & \text{if } -1/2 < t < 1, \\ [0, n-1] & \text{if } t = 1, \\ (n-1)t^{n-2} & \text{if } 1 < t. \end{cases}$$

For every  $t \in K_1$ , the Euclidean gradient of  $f_2(t, \cdot)$  at  $X \in U_2 = M_n^+(\mathbf{R})$  is

$$f_2'(t, \cdot)(X) = -g(t)(t+1) \left( \text{tr}(X^{-1}) + 1 \right)^t X^{-2} + h(t)X^{-1},$$

thus the Riemannian derivative has the form

$$\begin{aligned} \partial_C^2 f_2(t, X) &= \text{grad} f_2(t, \cdot)(X) = X f_2'(t, \cdot)(X) X \\ &= -g(t)(t+1) \left( \text{tr}(X^{-1}) + 1 \right)^t I_n + h(t)X. \end{aligned}$$

The above expressions show that  $\mathbf{K} \ni (t, X) \mapsto \partial_C^\Delta \mathbf{f}(t, X)$  is upper semicontinuous. Therefore, Theorem 4.2 implies that  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$ , and its elements  $(\tilde{t}, \tilde{X}) \in \mathbf{K}$  are precisely the solutions of the system

$$\begin{cases} \xi_1(t - \tilde{t}) \geq 0 \text{ for some } \xi_1 \in \partial_C^1 f_1(\tilde{t}, \tilde{X}) \text{ for all } t \in K_1, \\ \langle \partial_C^2 f_2(\tilde{t}, \tilde{X}), \exp_{\tilde{X}}^{-1} Y \rangle_{\tilde{X}} \geq 0 & \text{for all } Y \in K_2, \end{cases} \quad (S'_2)$$

We notice that the solutions of  $(S'_2)$  and  $(S_2)$  coincide. In fact, we may show that  $\mathbf{f} \in \mathcal{K}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ ; thus from Theorem 3.1 (ii) we have that  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ . It is clear that the map  $t \mapsto f_1(t, X)$  is convex on  $U_1$  for every  $X \in K_2$ . Moreover,  $X \mapsto f_2(t, X)$  is also a convex function on  $U_2 = M_n^+(\mathbf{R})$  for every  $t \in K_1$ . Indeed, fix  $X, Y \in K_2$  and let  $\gamma_{X,Y}^H : [0, 1] \rightarrow K_2$  be the unique geodesic segment connecting  $X$  and  $Y$ , see (34). For every  $s \in [0, 1]$ , we have that

$$\begin{aligned} \ln \det(\gamma_{X,Y}^H(s)) &= \ln((\det X)^{1-s}(\det Y)^s) \\ &= (1-s) \ln \det X + s \ln \det Y \\ &= (1-s) \ln \det(\gamma_{X,Y}^H(0)) + s \ln \det(\gamma_{X,Y}^H(1)). \end{aligned}$$

The Riemannian Hessian of  $X \mapsto \text{tr}(X^{-1})$  with respect to  $\langle \langle \cdot, \cdot \rangle \rangle$  is

$$\text{Hess}(\text{tr}(X^{-1}))(V, V) = \text{tr}(X^{-2}VX^{-1}V) = |X^{-1}VX^{-1/2}|_F^2 \geq 0,$$

where  $|\cdot|_F$  denotes the standard Fröbenius norm. Thus,  $X \mapsto \text{tr}(X^{-1})$  is convex (see Udrişte [33, §3.6]), so  $X \mapsto (\text{tr}(X^{-1}) + 1)^{t+1}$ . Combining the above facts with the non-negativity of  $g$  and  $h$  (see (39)), it yields that  $\mathbf{f} \in \mathcal{K}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$  as we claimed.

By recalling the notation from (38), the inequality of Newton has the form

$$S_2(Y) \leq \frac{n-1}{2n} \text{tr}^2(Y) \text{ for all } Y \in M_n(\mathbf{R}). \quad (41)$$

The possible cases are as follow:

- (a) Let  $0 \leq \tilde{t} < 1$ . Then the first relation from  $(S'_2)$  implies  $-\frac{4nS_2(\tilde{X})}{\tilde{t}+1} \geq 0$ , a contradiction.
- (b) If  $1 < \tilde{t} < 2$ , the first inequality from  $(S'_2)$  holds if and only if

$$S_2(\tilde{X}) = \frac{n-1}{4n} \tilde{t}^{n-2}(\tilde{t}+1) \text{tr}^2(\tilde{X}),$$

which contradicts Newton's inequality (41).

- (c) If  $\tilde{t} = 2$ , from the first inequality of  $(S'_2)$  it follows that

$$3(n-1)2^{n-4} \text{tr}^2(\tilde{X}) \leq nS_2(\tilde{X}),$$

contradicting again (41).

(d) Let  $\tilde{t} = 1$ . From the first relation of  $(S'_2)$  we necessarily have that  $0 = \xi_1 \in \partial_C^1 f_1(1, \tilde{X})$ . This fact is equivalent to

$$\frac{2nS_2(\tilde{X})}{\text{tr}^2(\tilde{X})} \in [0, n-1],$$

which holds true, see (41). In this case, the second relation from  $(S'_2)$  becomes

$$-2g(1) \left( \text{tr}(\tilde{X}^{-1}) + 1 \right) \langle \langle I_n, \exp_{\tilde{X}}^{-1} Y \rangle \rangle_{\tilde{X}} + h(1) \langle \langle \tilde{X}, \exp_{\tilde{X}}^{-1} Y \rangle \rangle_{\tilde{X}} \geq 0, \quad \forall Y \in K_2.$$

By using (33) and the well-known formula  $e^{\text{tr}(\ln X)} = \det X$ , the above inequality reduces to

$$-2g(1)(\text{tr}(\tilde{X}^{-1}) + 1)\text{tr}(\tilde{X}^{-1} \ln(\tilde{X}^{-1/2} Y \tilde{X}^{-1/2})) + h(1) \ln \frac{\det Y}{\det \tilde{X}} \geq 0, \quad \forall Y \in K_2. \quad (42)$$

We also distinguish three cases:

(d1) If  $g(1) = h(1) = 0$ , then  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{1\} \times K_2$ .

(d2) If  $g(1) = 0$  and  $h(1) > 0$ , then (42) implies that  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{(1, \tilde{X}) \in \mathbf{K} : \det \tilde{X} = 1\}$ .

(d3) If  $g(1) > 0$ , then (39) implies that  $(1, I_n) \in \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ .  $\diamond$

**Remark 6.1** We easily observed in the case (d3) that  $\tilde{X} = I_n$  solves (42). Note that the same is not evident at all for the second inequality in  $(S_2)$ . We also notice that the determination of the whole set  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$  in (d3) is quite difficult; indeed, after a simple matrix-calculus we realize that (42) is equivalent to the equation

$$\tilde{X} = P_{K_2} \left( e^{-\frac{h(1)}{2g(1)(\text{tr}(\tilde{X}^{-1})+1)}} \tilde{X} e^{\tilde{X}^{-1}} \right),$$

where  $P_{K_2}$  is the metric projection with respect to the metric  $d_H$ .

**Example 6.3** Let

$$K_1 = [0, \infty), \quad K_2 = \{X \in M_n^+(\mathbf{R}) : \text{tr}(X^{-1}) \leq n\},$$

and the functions  $f_1, f_2 : K_1 \times K_2 \rightarrow \mathbf{R}$  defined by

$$f_1(t, X) = t^3 \det X - (t-1)\text{tr}(X^{-1}), \quad f_2(t, X) = g(t)\text{tr}(\ln^2 X) + h(t)\text{tr}(X^{-1}),$$

where  $g, h : K_1 \rightarrow \mathbf{R}$  are two continuous functions such that

$$\inf_{K_1} g > 0 \text{ and } h \text{ is bounded.} \quad (43)$$

Here,  $M_n^+(\mathbf{R})$  is endowed with the Riemannian metric  $\langle\langle \cdot, \cdot \rangle\rangle$  defined in (33). Note that  $K_2$  is geodesic convex in  $(M_n^+(\mathbf{R}), \langle\langle \cdot, \cdot \rangle\rangle)$  being a sub-level set of the convex function  $X \mapsto \text{tr}(X^{-1})$ , see Example 6.2. However,  $K_2$  is not compact in  $(M_n^+(\mathbf{R}), \langle\langle \cdot, \cdot \rangle\rangle)$ . Indeed, it is clear that  $X_k = kI_n \in K_2$  ( $k \geq 1$ ), but on account of (40), we have  $d_H(I_n, X_k) = \sqrt{n \ln k} \rightarrow \infty$  as  $k \rightarrow \infty$ .

By extending the functions  $f_1(\cdot, X)$  to  $\mathbf{R}$  and  $f_2(t, \cdot)$  to  $M_n^+(\mathbf{R})$  by the same expressions, it becomes clear that  $\mathbf{f} = (f_1, f_2) \in \mathcal{C}(\mathbf{K}, \mathbf{U}, \mathbf{M})$ , where  $\mathbf{U} = \mathbf{M} = \mathbf{R} \times M_n^+(\mathbf{R})$  is the standard product manifold with metric  $\mathbf{g}$ , see (11). Although the functions  $X \mapsto \text{tr}(\ln^2 X)$  and  $X \mapsto \text{tr}(X^{-1})$  are convex,  $\mathbf{f} = (f_1, f_2)$  does not necessarily belong to  $\mathcal{K}(\mathbf{K}, \mathbf{U}, \mathbf{M})$  since  $h$  can be sign-changing. A simple calculation based on (40) and (5) gives that

$$\partial_C^\Delta \mathbf{f}(t, X) = (3t^2 \det X - \text{tr}(X^{-1}), -2g(t) \exp_X^{-1} I_n - h(t)I_n),$$

which is a continuous map on  $\mathbf{K}$ .

In order to apply Theorem 4.3, we will verify hypothesis  $(H_{\mathbf{p}_0})$  with  $\mathbf{p}_0 = (0, I_n) \in \mathbf{K}$ . Fix  $\mathbf{p} = (t, X) \in \mathbf{K}$  arbitrarily. Then, we have

$$\langle \partial_C^\Delta \mathbf{f}(\mathbf{p}_0), \exp_{\mathbf{p}_0}^{-1}(\mathbf{p}) \rangle_{\mathbf{g}} = -nt - h(0) \langle \langle I_n, \exp_{I_n}^{-1} X \rangle \rangle_{I_n} = -nt - h(0) \ln \det X,$$

while by (4) and (40), we also obtain

$$\begin{aligned} \langle \partial_C^\Delta \mathbf{f}(\mathbf{p}), \exp_{\mathbf{p}}^{-1}(\mathbf{p}_0) \rangle_{\mathbf{g}} &= -t(3t^2 \det X - \text{tr}(X^{-1})) \\ &\quad + \langle \langle -2g(t) \exp_X^{-1} I_n - h(t)I_n, \exp_X^{-1} I_n \rangle \rangle_X \\ &= -3t^3 \det X + t \text{tr}(X^{-1}) \\ &\quad - 2g(t) \text{tr}(\ln^2 X) - h(t) \langle \langle I_n, \exp_X^{-1} I_n \rangle \rangle_X. \end{aligned}$$

On the one hand, by (2), (4), and the inequality  $\text{tr}(A^2) \leq \text{tr}^2(A)$  (for  $A \in M_n^+(\mathbf{R})$ ), we have

$$\begin{aligned} |\langle \langle I_n, \exp_X^{-1} I_n \rangle \rangle_X| &\leq \|I_n\|_X \|\exp_X^{-1} I_n\|_X = (\text{tr}(X^{-2}))^{1/2} d_H(I_n, X) \\ &\leq \text{tr}(X^{-1}) d_H(I_n, X) \leq n d_H(I_n, X) = n(\text{tr}(\ln^2 X))^{1/2}. \end{aligned}$$

On the other hand, since  $X \in K_2$ , one has

$$(\det X^{-1})^{1/n} \leq \frac{\text{tr}(X^{-1})}{n} \leq 1,$$

thus,  $\det X \geq 1$ . On account of (43) and the above estimations, one has

$$\frac{-3t^3 \det X - 2g(t) \text{tr}(\ln^2 X) - nt + t \text{tr}(X^{-1}) - h(0) \ln \det X - h(t) \langle \langle I_n, \exp_X^{-1} I_n \rangle \rangle_X}{(t^2 + \text{tr}(\ln^2 X))^{1/2}} \rightarrow -\infty,$$

as  $\mathbf{d}_{\mathbf{g}}(\mathbf{p}, \mathbf{p}_0) = (t^2 + \text{tr}(\ln^2 X))^{1/2} \rightarrow \infty$ . The latter limit and the above expressions show that hypothesis  $(H_{\mathbf{p}_0})$  holds true with  $L_{\mathbf{p}_0} = -\infty$ . Now, we are in the position to apply Theorem 4.3, i.e.,  $\mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) \neq \emptyset$ , while its elements  $(\tilde{t}, \tilde{X}) \in \mathbf{K}$  are the solutions of the system

$$\begin{cases} (3\tilde{t}^2 \det \tilde{X} - \text{tr}(\tilde{X}^{-1}))(t - \tilde{t}) \geq 0 & \text{for all } t \in K_1, \\ \langle \langle -2g(\tilde{t}) \exp_{\tilde{X}}^{-1} I_n - h(\tilde{t}) I_n, \exp_{\tilde{X}}^{-1} Y \rangle \rangle_{\tilde{X}} \geq 0 & \text{for all } Y \in K_2. \end{cases} \quad (S_3)$$

Let us assume that  $\tilde{t} = 0$ ; then from the first relation of  $(S_3)$  we necessarily obtain  $-\text{tr}(\tilde{X}^{-1}) \geq 0$ , a contradiction. Thus,  $\tilde{t} > 0$ ; in particular, from the first relation of  $(S_3)$  it yields that

$$3\tilde{t}^2 \det \tilde{X} - \text{tr}(\tilde{X}^{-1}) = 0. \quad (44)$$

Since  $\tilde{X} \in K_2$ , the latter relation implies that

$$3\tilde{t}^2 = \text{tr}(\tilde{X}^{-1}) \det \tilde{X}^{-1} \leq \text{tr}(\tilde{X}^{-1}) \left( \frac{\text{tr}(\tilde{X}^{-1})}{n} \right)^n \leq n.$$

This estimate gives the idea to distinguish the following two cases:

(a) Assume  $h(\sqrt{\frac{n}{3}}) \leq 0$ . We may choose  $\tilde{t} = \sqrt{\frac{n}{3}}$  and  $\tilde{X} = I_n$  in  $(S_3)$ , taking into account that  $\langle \langle I_n, \exp_{I_n}^{-1} Y \rangle \rangle_{I_n} = \ln \det Y \geq 0$  for every  $Y \in K_2$ . Consequently,  $(\sqrt{\frac{n}{3}}, I_n) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ . A direct computation also shows that  $(\sqrt{\frac{n}{3}}, I_n) \in \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ .

(b) Assume  $h(\sqrt{\frac{n}{3}}) > 0$ . We define the function  $j : (0, \sqrt{\frac{n}{3}}] \rightarrow [1, \infty)$  by  $j(t) = \left( \sqrt{\frac{n}{3}} t^{-1} \right)^{\frac{1}{n+1}}$ . Since  $\lim_{t \rightarrow 0^+} j(t) = +\infty$ ,  $j(\sqrt{\frac{n}{3}}) = 1$ , relation (43), the continuity of the functions  $g, h, j$ , and our assumption imply that the equation

$$j(t) \ln j(t) = \frac{h(t)}{2g(t)} \quad (45)$$

has at least a solution  $\tilde{t}$  with  $0 < \tilde{t} < \sqrt{\frac{n}{3}}$ . We claim that  $(\tilde{t}, \tilde{X}) = (\tilde{t}, j(\tilde{t})I_n) \in \mathbf{K}$  solves  $(S_3)$ , i.e.,  $(\tilde{t}, j(\tilde{t})I_n) \in \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K})$ . First of all, we have that  $\tilde{X} = j(\tilde{t})I_n \in K_2$ ; indeed,  $\text{tr}(\tilde{X}^{-1}) = (j(\tilde{t}))^{-1}n < n$ . Then, a simple calculation shows that  $(\tilde{t}, j(\tilde{t})I_n)$  verifies (44). It remains to verify the second relation in  $(S_3)$ ; to complete this fact, a direct calculation and (45) give that

$$\langle \langle \exp_{\tilde{X}}^{-1} I_n + \frac{h(\tilde{t})}{2g(\tilde{t})} I_n, \exp_{\tilde{X}}^{-1} Y \rangle \rangle_{\tilde{X}} = j(\tilde{t}) \left( \frac{h(\tilde{t})}{2g(\tilde{t})} - j(\tilde{t}) \ln j(\tilde{t}) \right) \langle \langle I_n, \exp_{\tilde{X}}^{-1} Y \rangle \rangle_{\tilde{X}} = 0,$$

which concludes our claim.

We also have that  $(\tilde{t}, j(\tilde{t})I_n) \in \mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ . This fact can be proved either by a direct verification based on matrix-calculus or by observing that  $X \mapsto f_2(\tilde{t}, X)$  is convex (since  $h(\tilde{t}) > 0$ ).  $\diamond$

**Remark 6.2** It is clear that (45) can have multiple solutions which provide distinct Nash(-Stampacchia) equilibria for  $(\mathbf{f}, \mathbf{K})$ .

**Example 6.4** (a) Assume that  $K_i$  is closed and convex in the Euclidean space  $(M_i, g_i) = (\mathbf{R}^{m_i}, \langle \cdot, \cdot \rangle_{\mathbf{R}^{m_i}})$ ,  $i \in \{1, \dots, n\}$ , and let  $\mathbf{f} \in \mathcal{C}_{(\mathbf{K}, \mathbf{U}, \mathbf{R}^m)}$  where  $m = \sum_{i=1}^n m_i$ . If  $\partial_C^\Delta \mathbf{f}$  is  $L$ -globally Lipschitz and  $\kappa$ -strictly monotone on  $\mathbf{K} \subset \mathbf{R}^m$ , then the function  $\mathbf{f}$  verifies  $(H_{\mathbf{K}}^{\alpha, \rho})$  with  $\alpha = \frac{\kappa}{L^2}$  and  $\rho = \frac{\kappa^2}{2L^2}$ . (Note that the above facts imply that  $\kappa \leq L$ , thus  $0 < \rho < 1$ .) Indeed, for every  $\mathbf{p}, \mathbf{q} \in \mathbf{K}$  we have that

$$\mathbf{d}_{\mathbf{g}}^2(\exp_{\mathbf{p}}(-\alpha \partial_C^\Delta \mathbf{f}(\mathbf{p})), \exp_{\mathbf{q}}(-\alpha \partial_C^\Delta \mathbf{f}(\mathbf{q})))$$

$$\begin{aligned} &= \|\mathbf{p} - \alpha \partial_C^\Delta \mathbf{f}(\mathbf{p}) - (\mathbf{q} - \alpha \partial_C^\Delta \mathbf{f}(\mathbf{q}))\|_{\mathbf{R}^m}^2 = \|\mathbf{p} - \mathbf{q} - (\alpha \partial_C^\Delta \mathbf{f}(\mathbf{p}) - \alpha \partial_C^\Delta \mathbf{f}(\mathbf{q}))\|_{\mathbf{R}^m}^2 \\ &= \|\mathbf{p} - \mathbf{q}\|_{\mathbf{R}^m}^2 - 2\alpha \langle \mathbf{p} - \mathbf{q}, \partial_C^\Delta \mathbf{f}(\mathbf{p}) - \partial_C^\Delta \mathbf{f}(\mathbf{q}) \rangle_{\mathbf{R}^m} + \alpha^2 \|\partial_C^\Delta \mathbf{f}(\mathbf{p}) - \partial_C^\Delta \mathbf{f}(\mathbf{q})\|_{\mathbf{R}^m}^2 \\ &\leq (1 - 2\alpha\kappa + \alpha^2 L^2) \|\mathbf{p} - \mathbf{q}\|_{\mathbf{R}^m}^2 = \left(1 - \frac{\kappa^2}{L^2}\right) \mathbf{d}_{\mathbf{g}}^2(\mathbf{p}, \mathbf{q}) \\ &\leq (1 - \rho)^2 \mathbf{d}_{\mathbf{g}}^2(\mathbf{p}, \mathbf{q}). \end{aligned}$$

(b) Let

$$K_1 = [0, \infty), \quad K_2 = \{X \in M_n(\mathbf{R}) : \text{tr}(X) \geq 1\},$$

and the functions  $f_1, f_2 : K_1 \times K_2 \rightarrow \mathbf{R}$  defined by

$$f_1(t, X) = g(t) - c \text{tr}(X), \quad f_2(t, X) = \text{tr}((X - h(t)A)^2).$$

Here,  $g, h : K_1 \rightarrow \mathbf{R}$  are two functions such that  $g$  is of class  $C^2$  verifying

$$0 < \inf_{K_1} g'' \leq \sup_{K_1} g'' < \infty, \tag{46}$$

$h$  is  $L_h$ -globally Lipschitz, while  $A \in M_n(\mathbf{R})$  and  $c > 0$  are fixed such that

$$c + L_h \sqrt{\text{tr}(A^2)} < 2 \inf_{K_1} g'' \quad \text{and} \quad cn + 2L_h \sqrt{\text{tr}(A^2)} < 4. \tag{47}$$

Now, we consider the space  $M_n(\mathbf{R})$  endowed with the inner product defined in (31). We observe that  $K_2$  is geodesic convex but not compact in  $(M_n(\mathbf{R}), \langle \cdot, \cdot \rangle)$ . After a natural extension of functions  $f_1(\cdot, X)$  to  $\mathbf{R}$  and  $f_2(t, \cdot)$  to the whole  $M_n(\mathbf{R})$ , we can state that  $\mathbf{f} = (f_1, f_2) \in \mathcal{C}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ , where  $\mathbf{U} = \mathbf{M} = \mathbf{R} \times M_n(\mathbf{R})$ . On account of (46), after a computation it follows that the map

$$\partial_C^\Delta \mathbf{f}(t, X) = (g'(t) - c \text{tr}(X), 2(X - h(t)A))$$

is  $L$ -globally Lipschitz and  $\kappa$ -strictly monotone on  $\mathbf{K}$  with

$$L = \max \left( (2 \sup_{K_1} g'' + 8L_h \operatorname{tr}(A^2))^{1/2}, (2c^2n + 8)^{1/2} \right) > 0,$$

$$\kappa = \min \left( \inf_{K_1} g'' - \frac{c}{2} - \frac{L_h \sqrt{\operatorname{tr}(A^2)}}{2}, 1 - \frac{cn}{4} - \frac{L_h \sqrt{\operatorname{tr}(A^2)}}{2} \right) > 0.$$

According to (a),  $\mathbf{f}$  verifies  $(H_{\mathbf{K}}^{\alpha, \rho})$  with  $\alpha = \frac{\kappa}{L^2}$  and  $\rho = \frac{\kappa^2}{2L^2}$ . On account of Theorem 4.4, the set of Nash-Stampacchia equilibrium points for  $(\mathbf{f}, \mathbf{K})$  contains exactly one point  $(\tilde{t}, \tilde{X}) \in \mathbf{K}$  and the orbits of both dynamical systems  $(DDS)_{\alpha}$  and  $(CDS)_{\alpha}$  exponentially converge to  $(\tilde{t}, \tilde{X})$ . Moreover, one also has that  $\mathbf{f} \in \mathcal{K}_{(\mathbf{K}, \mathbf{U}, \mathbf{M})}$ ; thus, due to Theorem 3.1 (ii) we have that  $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K}) = \mathcal{S}_{NS}(\mathbf{f}, \mathbf{K}) = \{(\tilde{t}, \tilde{X})\}$ .  $\diamond$

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